Deliberation and the Wisdom of Crowds

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– incomplete[1] draft –

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Abstract

Does group deliberation increase group competence, measured by the correctness probability of majority decisions? We present a model of opinion formation based on sources, and a non-game-theoretic model of deliberation as sharing. Two jury theorems, one pre-deliberation and one post-deliberation, suggest that deliberation can improve group decisions. Three major failures of voting outcomes are: (1) overresponse to widespread evidence, when different evidence has spread to different extents across voters; (2) neglect of the strength of voter evidence, as all voters have equal weight; (3) neglect of informational complementarities, i.e., failure to use information that would follow by combining evidence of different voters. Partly drawing on simulations, we argue that deliberation normally reduces all three failures. But there exist systematic exceptions where deliberation increases Failure 1, sometimes even to the extent that group competence overall falls. Our findings suggest a recommendation for 'balanced' deliberation, which neither privileges certain individuals nor privileges certain evidence.

1 Introduction

Group deliberation tends to improve group decisions. A growing body of work argues informally for this thesis, sometimes also warning of 'traps' where deliberation is harmful. Yet the formal understanding of the collective merits of deliberation is at an early and disjointed stage. The lack of formal understanding of the deliberation process is particularly striking when compared to a different process, that of aggregation or voting, whose epistemic merits are studied extensively in the literature about jury theorems.

This paper presents a formal analysis of deliberation as evidence sharing. Although our notion of evidence is entirely broad and includes empirical evidence as much as theoretical arguments and other relevant insights or considerations, real deliberation goes clearly beyond evidence sharing, by covering dimensions such as:

[1] The literature review, three appendices, and the list of references are missing, and the main text needs some extensions and revisions. Meanwhile, details are available on request from the authors.

discovery of entirely new evidence, design of initially unimagined choice options, and change of individual preferences, typically towards more other-regarding preferences. We set those other aspects aside, by presupposing that the options are given and that the preferences are already aligned and 'truth-oriented'. Specifically, we presuppose a classic epistemic choice problem: one option is 'correct' in an objective (or intersubjective) sense, and all group members would like the correct option to be chosen, but hold different opinions about which option is correct, due to holding different evidence. We assume that the group makes its choice by majority voting, and ask whether deliberation prior to voting improves the voting outcome.

In principle, one would hope that the (majority) voting outcome selects the option that is most likely correct given the totality of evidence spread across its members. This can fail to happen, for at least three reasons:

- *Failure 1: overrespecting widespread evidence.* The voting outcome effectively 'contains' each evidence as many times as it is 'contained' in a vote, i.e., as there are voters who access that evidence. Widespread evidence is multicounted, hence overappreciated, while private evidence is counted just once, hence underappreciated, although the spread of evidence is epistemically irrelevant.
- *Failure 2: neglecting the strength of voter evidence.* Voting outcomes count voters equally, despite unequally strong evidence. While an equal treatment of all voters can be defended on grounds of fairness or equally, it is epistemically inefficient when evidence strength differs.
- *Failure 3: neglecting informational complementarities.* The voting outcome fails to 'contain' knowledge that follows from the combination of evidence of different voters, without already being 'contained' in the evidence of any single voter. Indeed, such knowledge is not 'contained' in any vote, and hence gets neglected.

All three failures stem from bad management of evidence that is in principle available. The hope is that deliberation can improve the use of information. Specifically:

- Deliberation might reduce Failure 1 by increasing the spread of previously private or almost private evidence. This reduces spread imbalance.
- Deliberation might reduce Failure 2 by letting voters with initially weak evidence accumulate evidence. This reduces interpersonal imbalance.
- Deliberation might reduce Failure 3 by letting individuals collect evidence from different members, and then use evidential complementarities.

But are these conjectures correct? What is the overall impact of deliberation on group competence? These questions have no obvious answer; for instance, deliberation could increase rather than reduce Failures 1 and 2.

This paper presents a framework for addressing these questions, and offers some analytic and simulation-based results. On the analytic side, we prove a jury theorem that sets an upper bound to collective competence, and we decompose the group's 'competence gap' into two gaps, one that deliberation can potentially close and another one that an increase of the group size can potentially close. Our simulations suggest that deliberation is much better at reducing Failure 2 than at reducing Failure 1, and support a recommendation for *balanced* forms of deliberation, in which all evidence is treated symmetrically and all group members participate equally. We finish the paper with more speculative considerations about deliberation, including the potential to reduce Failure 3 and the potential to counterbalance 'noise' or 'irrationalities' affecting personal opinions.

2 A model of opinions and deliberation

This section presents our model of opinion formation and deliberation, in the simple version that will later form the basis of our simulations. The generalised model is presented later.

2.1 Opinions and their sources

A group of persons labelled $1, ..., n$ faces two options labelled 1 and -1. The group is denoted $N = \{1, ..., n\}$ and can have any finite size $n \geq 1$. Following the epistemic paradigm, exactly one option is objectively or intersubjectively *correct*; it is called the *state of the world*, for short the *state*. We represent it by a random variable $\mathbf{x}$ taking the value 1 or -1. In general, we denote random variables in bold and their particular values in non-bold, and we take all random variables and events to be defined relative to a fixed probability space, with probability measure Pr.

Each person forms an opinion about which opinion is correct. There are three possible opinions: the opinion that option 1 is correct (labelled '1'), the opinion that option -1 is correct (labelled '-1'), and a neutral or undecided opinion (labelled '0'). Opinions are based on evidences (the generalised model presented later also allows non-evidential influences). Formally, let S be a finite non-empty set of *sources (of evidence)*, and for each source $s \in S$ let $\mathbf{e}_s$ be a real-valued random variables, the *evidence* from source s. A positive, negative, or zero value of an evidence represents support for option 1, support for option -1, or evidential neutrality, respectively. The absolute value of the evidence indicates the strength of support. If the source s is a witness report, the evidence $\mathbf{e}_s$ measures which opinion is supported by the report, and how strongly.

Each person i accesses some set of sources, called her *source set* and represented by a random variable $\mathbf{S}_i$ whose values are subsets of S. In a court jury, a juror's source set might contain some witness report and some legal text interpreting the law, while another juror's source set might contain the defendant's facial expression and other sources of evidence.

We can now define the concepts of personal opinion, personal competence, majority opinion, and majority competence. The *opinion of person* i is the option supported by i's total evidence:

$$\mathbf{o}_i = \begin{cases} 1 & \text{if } \sum_{s \in \mathbf{S}_i} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s \in \mathbf{S}_i} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s \in \mathbf{S}_i} \mathbf{e}_s = 0. \end{cases}$$

The *competence* of a person i is the probability of a correct opinion $p_i = Pr(\mathbf{o}_i = \mathbf{x})$. The *majority opinion* depends on whether more persons have opinion 1 or opinion -1, formally:

$$\mathbf{o}_{maj} = \begin{cases} 1 & \text{if } |\{i : \mathbf{o}_i = 1\}| > |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i > 0 \\ -1 & \text{if } |\{i : \mathbf{o}_i = 1\}| < |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i < 0 \\ 0 & \text{if } |\{i : \mathbf{o}_i = 1\}| = |\{i : \mathbf{o}_i = -1\}|, \text{ equivalently } \sum_i \mathbf{o}_i = 0. \end{cases}$$

The *majority competence* is the probability of a correct majority opinion $p_{maj} = P(\mathbf{o}_{maj} = \mathbf{x})$.

[HERE OR EARLIER A CAUSAL DIAGRAM LIKE IN OUR JURY ENTRY]

Diversity within the group can be construed as heterogeneity in sources, i.e., dissimilarity between the source sets $\mathbf{S}_i$ of different persons i. Under minimal diversity, people have identical source sets, so identical opinions. Under maximal diversity, they have pairwise disjoint source sets, hence state-conditionally independent opinions. A different concept is that of *intrapersonal diversity.* Someone has high intrapersonal diversity if they have a large source set, hence an opinion with a broad basis. As will emerge, deliberation tends to 'internalise' diversity, by letting sources be more widely shared, which transforms interpersonal into intrapersonal diversity.

The *source profile* is the combination of source sets across people $(\mathbf{S}_i)_{i \in N}$, in short $(\mathbf{S}_i)$. Person i's *evidence bundle* is the family of evidences from her sources $(\mathbf{e}_s)_{s \in \mathbf{S}_i}$; it is doubly random, through her source set $\mathbf{S}_i$ and the evidences $\mathbf{e}_s$ from her sources s. The *evidence profile* is the combination of evidence bundles across people $((\mathbf{e}_s)_{s \in \mathbf{S}_i})_{i \in N}$, in short $((\mathbf{e}_s)_{s \in \mathbf{S}_i})$.

We make three simplifying assumptions in the model underlying our simulations (they will be dropped later in the generalised model).

Equiprobable States: the state $\mathbf{x}$ takes both values 1 and -1 with probability $\frac{1}{2}$.

Simple Gaussian Evidences: Given any state $x \in \{\pm 1\}$, the evidences $\mathbf{e}_s$ $(s \in S)$ have independent Gaussian distributions with mean x and some variance σ^2 that is the same across states x and sources s. So, each evidence correlates positively with the state: positive evidence objectively supports state 1, negative evidence objectively supports state -1. This positive correlation is the rationale behind our definition of opinions $\mathbf{o}_i$, in which each evidence pulls the opinion towards the state that has the same sign as the evidence.[2]

Independent Sources: The source-access events are independent across people and sources, and jointly independent of the state and the evidences. Formally, for each person i and source $s \in S$ we consider the event that person i accesses source s, i.e., that $s \in \mathbf{S}_i$, and we require that these 'source-access events' be mutually independent, and jointly independent of the state-evidence combination $(\mathbf{x}, (\mathbf{e}_s)_{s \in S})$. For instance, when forming opinions about tomorrow's weather (the state), whether person 1 has listened to ('accesses') the weather forecast is independent of which other sources she and others access, and also independent of tomorrow's weather (the state) and the evidence from each source (e.g., from the prediction made by the weather forecast).

The probability that person i accesses source s will be denoted $p_{s \to i} = Pr(s \in \mathbf{S}_i)$, and called an *access probability*. The family of access probabilities $p_{s \to i}$ fully determines the distribution of the source profile $(\mathbf{S}_i)$. How? We use Independent Sources twice. First, the probability that person i has source set S_i is the product of the probabilities of accessing any source in S_i and not accessing any other source:

$$Pr(S_i) = \left(\prod_{s \in S_i} p_{s \to i} \right) \left(\prod_{s \in S \setminus S_i} \overline{p_{s \to i}} \right) \text{ for each person } i. \tag{1}$$

Second, the probability of an entire source profile (S_i) is the product $\prod_i Pr(S_i)$, with $Pr(S_i)$ given by (1).

To summarise, our formal primitive is a *simple opinion structure*, by which we mean a triple $(\mathbf{x}, (\mathbf{e}_s)_{s \in S}, (\mathbf{S}_i)_{i \in N})$, in short $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, containing:

(1) a random variable $\mathbf{x}$, the *state* or *correct option*, taking the value 1 or -1 with equal probability;

(2) a family $(\mathbf{e}_s)$, indexed by some set S of *sources* (non-empty and finite), consisting of real-valued random variables, the *evidences* from these sources, which have state-conditionally independent Gaussian distributions with mean $\mathbf{x}$ and with some fixed variance $\sigma^2 > 0$;

(3) a family $(\mathbf{S}_i)$, indexed by some set $N = \{1, ..., n\}$ of *persons* $(1 \leq n < \infty)$, consisting of random subsets of S, the *source sets* of these persons, with distributions determined by access probabilities $(p_{s \to i})_{s \in S, i \in N}$ via (1), independently across persons and independently of the state and the evidences.

[2]Under generalised Gaussian assumptions, the evidences are possibly dependent (given the state), with means and/or variances that can vary across states and/or sources.

2.2 The rationality of opinions

Is this opinion model ad hoc from a rationality perspective? The worry is natural, as we presuppose a seemingly naive rationale for forming opinions: adding up one's evidences and comparing the sum with zero. In fact, such opinion formation *is* rational in a perfectly classical sense. Why?

Classic rationality requires evaluating options by expected utility. Given our epistemic setting, let us identify 'utility' with 'correctness level', defined as 1 if the opinion is correct, 0 if it is incorrect, and $\frac{1}{2}$ if it is neutral, i.e., zero. To be rational, the opinion of a person i should beat any possible alternative opinion, i.e., any alternative response to her information. Her information is her evidence bundle $(\mathbf{e}_s)_{s\in\mathbf{S}_i}$, which tells what her sources are (the value of $\mathbf{S}_i$) and what the evidences from these sources are (the value of $\mathbf{e}_s$ for each $s \in \mathbf{S}_i$). A *possible opinion* of person i is thus any random variable $\mathbf{o}$ that generates 1, -1 or 0 depending solely on the information $(\mathbf{e}_i)_{i\in\mathbf{S}_i}$. Its *correctness level* is 1 if $\mathbf{o} = \mathbf{x}$ (correct opinion), 0 if $\mathbf{o} = -\mathbf{x}$ (false opinion), and $\frac{1}{2}$ if $\mathbf{o} = 0$ (neutral opinion). Person i or her opinion $\mathbf{o}_i$ is *classically rational* if the expected correctness level of $\mathbf{o}_i$ weakly exceeds that of any possible opinion $\mathbf{o}$.

Theorem 1 *Under any simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, *the opinion* $\mathbf{o}_i$ *of any person* i *is classically rational.*

By contrast, non-simple opinion structures (Section 5) will allow for boundedly rational opinion formation. We note that our definition of rationality already applies to general opinion structures.

2.3 Deliberation as sharing

We construe deliberation as a process of sharing sources. We here restrict attention to *share-absorb processes*; general deliberation processes are defined later. A share-absorb process is given by parameters of two types, namely for each source $s \in S$ and person $i \in N$ a 'sharing probability' $p_{s,i\rightarrow}$ and a 'absorbing probability' $p_{s,i\leftarrow}$, both in $[0, 1]$. The process transforms the initial source profile $(\mathbf{S}_i)$ into a post-deliberation source profile $(\mathbf{S}_i^+)$, in two steps. First, each person i communicates ('shares') each of her initial sources $s \in \mathbf{S}_i$ with an independent probability of $p_{s,i\rightarrow}$. Second, for each source s shared by at least someone, each person i with $s \notin \mathbf{S}_i$ acquires ('absorbs') this source with an independent probability of $p_{s,i\leftarrow}$. This produces a post-deliberation source profile $(\mathbf{S}_i^+)$ in which a person i's new source set $\mathbf{S}_i^+$ contains i's initial sources in $\mathbf{S}_i$ *and* all sources shared by someone and absorbed by i. The sharing probability $p_{s,i\rightarrow}$ may be source-dependent, for instance because sources may be easier or harder to communicate; it may be person-dependent, for instance because persons may be more or less able or willing to communicate. The absorbing probability $p_{s,i\leftarrow}$ may also vary across sources or people. For instance, people may have different abilities to perceive evidence or arguments, partly due to different perspectives, conceptual frames, attention levels, or biases. An entirely different reason for a low – in fact, *zero* – absorbing probability obtains if deliberation is organised in subgroups, so that messages are shared and absorbed only within subgroups.

A share-absorb process therefore generates a new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$, in which individuals draw on richer sources. Like the initial opinion structure, the new one induces derivative constructs, namely opinions, competence levels, and (as will soon be seen) imbalance measures. They are defined as usual, but based now on the new opinion structure; we denoted them by adding a superscript '+' to the usual symbol. In particular, all persons i form new opinions, given by

$$\mathbf{o}_i^+ = \begin{cases} 1 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s\in\mathbf{S}_i^+} \mathbf{e}_s = 0, \end{cases}$$

implying new competence levels p_i^+ $(= Pr(\mathbf{o}_i^+ = \mathbf{x}))$, a new group opinion $\mathbf{o}_{maj}^+$, and a new group competence p_{maj}^+ $(= Pr(\mathbf{o}_{maj}^+ = \mathbf{x}))$.

[HERE TWO CAUSAL DIAGRAMS SIDE BY SIDE, PRE- &
POST-DELIBERATION]

This machinery allows to compare different deliberation processes (with different sharing and absorbing probabilities) in terms of the quality of the resulting individual and collective opinions. For instance, one would hope that the process increases group competence, i.e., that $p_{maj}^+ > p_{maj}$.

In fact, the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ is an opinion structure in the general sense addressed later, as Independent Sources fails post-deliberation. But the departure from a simple structure is small enough that the rationality result (Theorem 1) still applies, as emerges from the proof.

Appendix B defines and describes share-absorb processes more formally.

2.4 The contrast to a game-theoretic deliberation model

One might contrast our deliberation model with a share-absorb *game*, i.e., the following dynamic game with incomplete information. *Stage 1*: nature randomly draws evidences $(e_s)_{s\in S}$ and personal source sets $(S_i)_{i\in N}$, where each person (player) i is informed of her evidence bundle $(e_s)_{s\in S_i}$ (her type). *Stage 2*: simultaneously, each person i chooses which subset of her source set S_i to share. *Stage 3:* simultaneously, each person i chooses which sources to absorb, among the sources that she did not acquire in Stage 1 and that someone else shared in Stage 2. Players' utility functions could be specified in different ways. Roughly, each player incurs a cost from sharing and absorbing, but (as good a deliberator) attaches positive value to possessing sources and to others possessing sources. One could add a *Stage 4*: simultaneously, everyone casts a vote in $\{1, -1, 0\}$. This turns the game into a share-absorb-*vote* game. Plausibly, player's preferences should be specified such that sharing and absorbing is costly, a correct voting outcome gives positive utility.

While important in its own right, we do not take the game-theoretic approach to deliberation, as it comes with a number of assumptions and commitments that we wish to avoid and that stand in a certain tension to informal theories of deliberation. Let us mention four points.

First, we are not committed to the instrumental ('consequentialist') motivations postulated by game-theoretic models. For comparison, note that game-theoretic models of voting are often criticised for ascribing instrumental motivations to voters, which neglects expressive voting and other forms of intrinsic motivation in which voters care about the act of voting rather than the voting outcome. Similarly, one may distinguish between instrumentally and intrinsically motivated deliberation, i.e., between caring about the outcome and caring about one's acts of sharing or absorbing.[3] Theorists of deliberation and deliberative democracy often invoke intrinsic motivations and non-strategic dispositions as part of the ethics (and to some extent the reality) of deliberation (cite Habermas?). We wish to allow both intrinsically and instrumentally motivated deliberators.

Second, we aim to allow boundedly rational deliberation. This issue is orthogonal to that of the nature of motivations: someone's deliberation behaviour can be more or less rational with respect to instrumental or to intrinsic motivations. Certainly, we do not exclude rationality. To maintain the rationality hypothesis, one should interpret the sharing probability $p_{s,i\rightarrow}$ (or absorbing probability $p_{s,i\leftarrow}$) as the probability that the circumstances make it rational for i to share s (or absorb s), where the 'circumstances' can contain rich contextual information, such as the level of background noise and the mood of the moment.

Third, a central aspect of deliberation is awareness growth, as opposed to mere information learning. We allow deliberators to lack imagination of what they do not know and *could* learn during deliberation: they may not know what they do not know. Technically, a person i need not know her set $S \backslash S_i$ of non-accessed sources. By contrast, the game-theoretic approach implicitly rests on the assumption that players have (common) knowledge of the game they play. Put differently, we allow

[3] In the above share-absorb game the outcome is the final knowledge distribution; in the above share-absorb-*vote* game it is the voting outcome. Players are assumed to have instrumental motivation, i.e., to derive utility from the outcome rather than own actions (setting aside personal costs or efforts, which also affect utility).

deliberation to enlarge someone's *awareness* – i.e., the knowledge (and conception) of possibilities – whereas a deliberation game presupposes that all events are anticipated as possibilities, so that everyone knows pre-deliberation what they *could* learn, what costs they *would* incur, what motivations guide the (strategic) behaviour of others, etc. Such anticipation is hard to reconcile with the notion of deliberation developed in political theory.

Fourth, in our model, (not) sharing and (not) absorbing sources need not be a matter of choice. Some sources in S_i may be impossible ('infeasible') to share for person i, e.g., due to language restrictions or even unawareness of holding the source. Similarly, someone may be unable to absorb ('reach', 'understand') some sources, because her perspective, conceptual scheme, or subconscious bias stand in the way.

3 The limited wisdom of crowds pre- and post-deliberation: two jury theorems

The wisdom of crowds is often defended through jury theorems. We now present two jury theorems – one pre-deliberation, one post-deliberation. Their message is less optimistic: the wisdom of crowds is objectively bounded, even in asymptotically large groups. This revisionary picture stems from common sources of different voters, which undermine the independence assumption of classical jury theorems.

Specifically, the group cannot perform better than the 'ideal opinion' – a hypothetical opinion based on total evidence. The ideal opinion is fallible since total evidence can lie. Worse, the group can fail to reach the ideal opinion, hence is even more fallible than the ideal opinion, by making bad use of the evidence scattered across its members.

Here deliberation steps in to help the group approach the ideal opinion, by making better use of the available information. This is suggested by our jury theorems. By contrast, classical jury theorems might have suggested that deliberation is inessential because sufficiently large groups find the truth anyway.

3.1 Pre-deliberation

Given our simple opinion structure, the *ideal opinion* is the opinion based on all sources, regardless of who (if anyone) accesses them:

$$\mathbf{o}_{IDEAL} = \begin{cases} 1 & \text{if } \sum_{s\in S} \mathbf{e}_s > 0 \\ -1 & \text{if } \sum_{s\in S} \mathbf{e}_s < 0 \\ 0 & \text{if } \sum_{s\in S} \mathbf{e}_s = 0. \end{cases}$$

The correctness probability of the ideal opinion is the *ideal competence* $p_{IDEAL} = Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$. By Appendix C, it is the probability that a standard normal variable takes a value below $\frac{\sqrt{|S|}}{\sigma}$, i.e.,

$$p_{IDEAL} = Pr(\mathbf{o}_{IDEAL} = \mathbf{x}) = F_{N(0,1)}\left(\frac{\sqrt{|S|}}{\sigma}\right), \tag{2}$$

where $F_{N(0,1)}$ is the standard-normal distribution function. The ideal competence is always below 1, and is increasing in the number of sources $|S|$ and decreasing in the evidence variance σ^2. For instance, it is $p_{IDEAL} \approx 0.868$ if $|S| = 5$ and $\sigma^2 = 4$.

Stating a jury theorem presupposes an obvious generalisation of the simple opinion structure $(\mathbf{x}, (\mathbf{e}_s)_{s\in S}, (\mathbf{S}_i)_{i\in N})$: the set of persons N is the infinite set $\{1, 2, ...\}$, called the 'population'. We then talk of a 'simple opinion structure *for an infinite population*'. In such a structure, we can consider groups $\{1, ..., n\} \subseteq N$ of any finite size $n \geq 1$, with a corresponding majority opinion denoted $\mathbf{o}_{maj}$ or more explicitly $\mathbf{o}_{maj,n}$, and a majority competence $Pr(\mathbf{o}_{maj,n} = \mathbf{x})$ denoted p_{maj} or more explicitly $p_{maj,n}$.

We now state our first jury theorem. It says that a finite group performs sub-ideally as long as people are not utterly perfect at accessing sources ('Imperfect Access'), but that an asymptotically large group reaches the ideal if people are sufficiently good at accessing sources ('Access Competence'). Formally:

Imperfect Access: At least one source $s \in S$ is not surely accessed, i.e., has access probability $p_{i\to s} < 1$ for each person i.

Access Competence: The probability $p_{s\to i}$ that a person $i \in N$ accesses a source $s \in S$ is at least $2^{-1/|S|} + \epsilon$, for some $\epsilon > 0$ independent of i and s.

Pre-deliberation Jury Theorem: *Given a simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ *for an infinite population, the majority competence* $p_{maj,n}$

(a) *is at most the ideal competence (2), and less than it under Imperfect Access,*

(b) *converges to it as* $n \to \infty$ *under Access Competence.*

Access Competence is hard to meet. For example, with $|S| = 5$ sources the access probability $p_{s\to i}$ must exceed $2^{-1/5} \approx 0.87$ for all persons i and sources s. Let us therefore turn to post-deliberation group opinions.

3.2 Post-deliberation

Now suppose the group deliberates before voting. So, consider a share-absorb process, again in a generalised sense with an infinite population $N = \{1, 2, ...\}$, defined by sharing and absorbing probabilities $(p_{s,i\to}, p_{s,i\leftarrow})_{s\in S, i\in N}$. For any finite group $\{1, ..., n\} \subseteq N$ (where $n \geq 1$), the process induces a standard share-absorb process for this group, defined by the subfamily of parameters $(p_{s,i\to}, p_{s,i\leftarrow})_{s\in S, i\in\{1,...,n\}}$ restricted to persons in $\{1, ..., n\}$. This (sub)process generates a post-deliberation source set $\mathbf{S}^+_{i,n}$ for each group member $i \in \{1, ..., n\}$, and hence a post-deliberation opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}^+_{i,n}))$, with personal opinions $\mathbf{o}^+_{i,n}$, personal competences $p^+_{i,n}$, a group opinion $\mathbf{o}^+_{i,n}$, and a group competence $p^+_{maj,n}$. All these concepts are defined as usual. The index 'n' signals the dependence on the current group size n. Crucially, the same person i can (and will) develop different post-deliberation opinions $\mathbf{o}^+_{i,n}$ depending on the size of the group that deliberates: the larger the group, the more sources are shared, hence absorbed. Hence personal post-deliberation opinions improve as the group grows: $p^+_{i,n} \leq p^+_{i,n+1} \leq p^+_{i,n+3} \leq \dots$ This does not automatically translate into a larger group competence $p^+_{maj,n}$, because the added group members may be less competent. Still, our post-deliberation theorem brings good news: majority opinions are asymptotically ideal under a far weaker competence condition than pre-deliberation. This weaker competence condition pertains not just to people's ability to initially access sources, but also to their ability to absorb sources during deliberation. We use the label 'acquisition' to refer to both phenomena, initial access and later reception:

Acquisition Competence: Informally, for all persons i and sources s, the person has a high access probability $p_{s\to i}$ or a high absorbing probability $p_{s,i\leftarrow}$ (or both). Formally, for all persons $i \in N$ and sources $s \in S$, the product $(1-p_{s\to i})(1-p_{s,i\leftarrow})$ is at most $1 - 2^{-1/|S|} - \epsilon$, for some $\epsilon > 0$ independent of i and s.

If people violate Access Competence because of too low access probabilities, they can still satisfy Acquisition Competence because their absorbing probabilities 'make up' for their low access probabilities. Intuitively, they get a second chance to acquire sources during deliberation. More formally:

Proposition 1 *Acquisition Competence is logically weaker than Access Competence.*

Proof. Assume Access Competence. To see why Acquisition Competence holds, note that for any $i \in N$ and $s \in S$ the inequality $(1 - p_{s\to i})(1 - p_{s,i\leftarrow}) \leq 1 - 2^{-1/|S|} - \epsilon$ is satisfied because $(1 - p_{s\to i})(1 - p_{s,i\leftarrow}) \leq 1 - p_{s\to i}$ and because by Access Competence $p_{s\to i} \geq 2^{-1/|S|} + \epsilon$. ■

Our result also uses a minimal form of 'sharing competence': new group members do not get fully unable to share sources in the limit. We now state this condition, followed by the theorem.

Non-vanishing Sharing Competence: For each source $s \in S$, the probability that person i accesses and then shares s, $p_{s\to i} \times p_{s,i\to}$, does not tend to 0 as $i \to \infty$.[4]

[4] While $p_{s,i\to}$ is the sharing probability *given* that the source was accessed, $p_{s\to i}p_{s,i\to}$ is the *unconditional* sharing probability, which is reduced by the possibility of not having accessed the source. Non-vanishing Sending Competence holds for instance if all $p_{s\to i}$ and all $p_{s,i\to}$ exceed some fixed lower bound $\epsilon > 0$.

Post-deliberation Jury Theorem: *Given a simple opinion structure* $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ *and a share-absorb process, both for an infinite population, the post-deliberation majority competence* $p^+_{maj,n}$

(a) *is at most the ideal competence (2), and less than it under Imperfect Access,*
(b) *converges to it as* $n \to \infty$ *under Acquisition Competence and Non-vanishing Sharing Competence.*

The upshot is that deliberation can lead to asymptotically ideal majority opinions even when people are arbitrarily bad at accessing sources (so that Access Competence fails), provided that during deliberation they are good enough at absorbing sources and at least minimally good at sharing sources (so that Acquisition Competence and Non-vanishing Sharing Competence hold).

3.3 Closing the competence gap: by deliberation or group increase?

Group competence usually falls short of ideal competence (2). The difference between ideal and actual group competence, $p_{IDEAL} - p_{maj}$, defines the *competence gap*. Two instruments can help reduce this gap: deliberation and group increase. How do both instruments operate and complement one another?

A source $s \in S$ is 'available' to the group if at least one member accesses it. Formally, the *available source set* is the union of personal source sets $\cup_{i=1}^n \mathbf{S}_i$. The *relatively ideal opinion* is the opinion based on the available source set $\cup_{i=1}^n \mathbf{S}_i$ rather than the full set S, denoted $\mathbf{o}_{ideal}$ or more explicitly $\mathbf{o}_{ideal,n}$, and defined like $\mathbf{o}_{IDEAL}$ but with 'S' replaced by '$\cup_i \mathbf{S}_i$'. Its correctness probability $Pr(\mathbf{o}_{ideal} = \mathbf{x})$ is the *relatively ideal competence*, denoted p_{ideal} or more explicitly $p_{ideal,n}$.

The competence gap $p_{IDEAL} - p_{maj}$ can now be decomposed into the sum of two gaps:

(1) the competence gap between the ideal and the relatively ideal opinion, $p_{IDEAL} - p_{ideal}$, which stems from the unavailability of some sources in S,
(2) the competence gap between the relatively ideal and the actual opinion, $p_{ideal} - p_{maj}$, which stems from imperfect use of available sources.

Deliberation is an attempt to close gap (2). It cannot close gap (1) because it does not 'discover' new sources (formally, because the new available set $\cup_{i=1}^n \mathbf{S}_i^+$ is no larger than the old one $\cup_{i=1}^n \mathbf{S}_i$ by definition of share-absorb process).[5] Gap (1) is instead closed by increasing group size – the room for maneuver in jury theorems. Indeed, p_{ideal} converges to p_{IDEAL} as $n \to \infty$, under the minimal assumption that the probability $p_{s \to i}$ of access to a source $s \in S$ does not converge

[5] Under a broader concept of deliberation, deliberation can also 'discover' sources that no one held initially, and thereby help close also gap (1). This would strengthen the case for deliberation further.

to 0 as $i \to \infty$. This assumption of 'non-vanishing access competence' guarantees that each source is ultimately accessed by *someone* when adding individuals.[6] Increasing the group can also affect gap (2); under the fortunate conditions of Access Competence it asymptotically closes gap (2) (in addition to gap (1)), so that deliberation is inessential asymptotically, as the Pre-deliberation Jury Theorem shows. But normally Access Competence fails, and deliberation is crucial for reducing gap (2), thereby rendering the group opinion more (relatively) ideal.

By the Post-deliberation Jury Theorem, the interplay of deliberation and group increase manages to fully close the competence gap asymptotically under interesting conditions. Going beyond ideal towards generally correct group opinions remains impossible, no matter how much the group deliberates or is increased, because of objectively limited evidence.

4 Does deliberation reduce Failures 1 and 2?

This section introduces imbalance indices that can serve as proxies of Failures 1 and 2, and then presents simulation results about the ability of deliberation to reduce these failures and to increase group competence. Failure 3 is addressed later.

4.1 Measuring imbalance in the group

Failures 1 and 2 each stem from an imbalance in the group – either an imbalance between (the spread of) sources or an imbalance between (the strength of evidence of) persons, respectively. To prepare the simulation of both failures, we now define two imbalance measures, which will later serve as proxies of Failures 1 and 2, respectively.

Spread imbalance. The (source-)spread imbalance is the average variation across sources of the spread of the source. How is it defined? The spread of a source s is the number of source owners, $\#\{i : s \in \mathbf{S}_i\}$. The absolute variation of spread between two distinct sources s and s' is $|\#\{i : s \in \mathbf{S}_i\} - \#\{i : s' \in \mathbf{S}_i\}|$. More relevant than the absolute variation is the 'relative' or 'percentage' variation. A change of spread from 1 to 3 persons and a change from 101 and 103 persons both represent the same absolute variation (by 2), but the first change represents a much larger *relative* variation. We calculate the relative variation of spread between sources s and s' by dividing the absolute variation by the average spread $\frac{1}{2}(\#\{i : s \in \mathbf{S}_i\} + \#\{i : s' \in \mathbf{S}_i\})$. (If both spreads are zero, we have divided 0 by 0; by convention, $\frac{0}{0} = 0$, here and elsewhere.)

[6] Formally, with probability one the available source set $\cup_{i=1}^{n} \mathbf{S}_i$ converges to the full set S as n increases.

Now the *spread imbalance* is defined as the average relative variation of spread across all pairs of distinct sources:

$$\begin{aligned}\mathbf{SI} &= \frac{1}{|S|\,(|S|-1)} \sum_{(s,s')\in S^2: s\neq s'} \text{‘spread imbalance between } s \text{ and } s'\text{’} \\ &= \frac{1}{|S|\,(|S|-1)} \sum_{(s,s')\in S^2: s\neq s'} \frac{|\#\{i : s\in \mathbf{S}_i\} - \#\{i : s'\in \mathbf{S}_i\}|}{\frac{1}{2}\left(\#\{i : s\in \mathbf{S}_i\} + \#\{i : s'\in \mathbf{S}_i\}\right)}.\end{aligned}$$

Note that $|S|\,(|S|-1)$ is the number of pairs of distinct sources (s,s'). The spread imbalance $\mathbf{SI}$ takes different values SI depending on the values S_i of the source sets $\mathbf{S}_i$.

Interpersonal imbalance. The interpersonal imbalance is the average variation across persons of evidence strength. The evidence strength of a person i is the absolute total evidence, $\left|\sum_{s\in\mathbf{S}_i} \mathbf{e}_s\right|$. The absolute variation of evidence strength between distinct persons i and j is $\left|\left|\sum_{s\in\mathbf{S}_i} \mathbf{e}_s\right| - \left|\sum_{s\in\mathbf{S}_j} \mathbf{e}_s\right|\right|$. What matters is, however, the ‘relative’ or ‘percentage’ variation of evidence strength between persons i and j. It is obtained by dividing the absolute variation by the average strength $\frac{1}{2}\left(\left|\sum_{s\in\mathbf{S}_i} \mathbf{e}_s\right| + \left|\sum_{s\in\mathbf{S}_j} \mathbf{e}_s\right|\right)$. The *interpersonal imbalance* is the average relative variation of evidence strength across all pairs of distinct individuals:

$$\begin{aligned}\mathbf{II} &= \frac{1}{n(n-1)} \sum_{(i,j)\in N^2: i\neq j} \text{‘imbalance between } i \text{ and } j\text{’} \\ &= \frac{1}{n(n-1)} \sum_{(i,j)\in N^2: i\neq j} \frac{\left|\left|\sum_{s\in\mathbf{S}_i} \mathbf{e}_s\right| - \left|\sum_{s\in\mathbf{S}_j} \mathbf{e}_s\right|\right|}{\frac{1}{2}\left(\left|\sum_{s\in\mathbf{S}_i} \mathbf{e}_s\right| + \left|\sum_{s\in\mathbf{S}_j} \mathbf{e}_s\right|\right)}.\end{aligned}$$

Here, $n(n-1)$ is the number of pairs of distinct persons (i,j). The interpersonal imbalance $\mathbf{II}$ takes different values II depending on the values of the evidence bundles $(\mathbf{e}_s)_{s\in\mathbf{S}_i}$.

Resulting imbalance versus systemic imbalance. $\mathbf{SI}$ and $\mathbf{II}$ measure *resulting or ex-post imbalance*, as a consequence of the form taken by each evidence $\mathbf{e}_s$ and each personal source set $\mathbf{S}_i$. Resulting imbalance should be contrasted with *systemic or ex-ante imbalance*, which consists in a tendency towards resulting imbalance. We measure systemic imbalance by expected resulting imbalance, and denote systemic spread imbalance by $\mathcal{SI} = \mathbb{E}(\mathbf{SI})$ and systemic interpersonal imbalance by $\mathcal{II} = \mathbb{E}(\mathbf{II})$. We shall often talk of ‘imbalance’ simpliciter, thereby referring either to systemic imbalance ($\mathcal{SI}$ and $\mathcal{II}$) or to resulting imbalance ($\mathbf{SI}$ and $\mathbf{II}$) – the context will leave no ambiguity.

4.2 The imbalance indices as proxies of Failures 1 and 2

Our simulations will use the two imbalance indices – spread imbalance and interpersonal imbalance – as proxies for the extent of Failures 1 and 2, respectively.

What justifies these proxies? Strictly speaking, Failure 1 does not consist in an imbalance in the group, but in a deficiency of collective outcomes caused by that imbalance; and similarly for Failure 2. This observation at first suggests measuring each failure as the reduction of majority competence caused by an imbalance of relevant type. Failure 1 would then be measured by the difference between hypothetical and actual group competence, where 'hypothetical' means 'after removing the imbalance of relevant type, ceteris paribus'.[7] However, hypothetical amendments of real outcomes with a 'ceteris paribus' clause are notoriously ambiguous and non-unique, as they involve a counterfactual construction. This is why we shall instead measure both failures by 'indirect' proxies, in the form of the two imbalance indices.

Just as imbalance can be understood as resulting or systemic imbalance, so Failures 1 and 2 (and 3) can be viewed resulting failures or as systemic failures (i.e., tendencies towards resulting failures). Understood as resulting failures, our proxies are **SI** and **II**. Understood as systemic failures, our proxies are $\mathcal{SI}$ and $\mathcal{II}$.

4.3 Setting the stage for the simulations

Our simulations work with share-absorb processes applied to simple opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. To identify phenomena and check robustness, we could in principle vary all parameters of the opinion structure and the process, i.e., the group size n, the source number $|S|$, the variance σ^2, the access probabilities $p_{s\to i}$, the sharing probabilities $p_{s,i\to}$, and the absorbing probabilities $p_{s,i\leftarrow}$. In fact, we will only vary the parameters $p_{s\to i}$, $p_{s,i\to}$ and $p_{s,i\leftarrow}$. As for n and $|S|$, our reference case is one with $n = 9$ voters and $|S| = 5$ sources. This case is rich enough for meaningful aggregation but limited enough to inspect results visually and to keep the computational costs low. We set the standard deviation of (Gaussian) evidence to $\sigma = 2$ for an intermediate quality of evidence.

All derivative concepts and their notation apply. Of particular interest in simulations will be the generated new source profile $(\mathbf{S}_i^+)$, the old and new group competence p_{maj} and p_{maj}^+, the old and new (systemic) imbalance indices $\mathcal{SI}$, $\mathcal{II}$, $\mathcal{SI}^+$, and $\mathcal{II}^+$.

To obtain numerical estimates of the epistemic merits of different forms of deliberation, we employ Monte Carlo simulations with 100,000 rounds each, and calculate average-based estimates of group competence and of both (systemic)

[7] Failure 1 at the pre-deliberation time would be measured by $\tilde{p}_{maj} - p_{maj}$, where $\tilde{p}_{maj}$ is the majority opinion relative to a hypothetical opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\tilde{\mathbf{S}}_i))$ with a source profile $(\tilde{\mathbf{S}}_i)$ that has been modified such that the spread imbalance is zero, ceteris paribus; and similarly for the measurement of Failure 2 at the pre-deliberation time, and of Failures 1 and 2 at the post-deliberation time.

imbalance indices.

4.4 Balanced deliberation

We start by investigating balanced settings. Table 1 reports a selection of results (a more comprehensive results table is in the appendix). The access probability

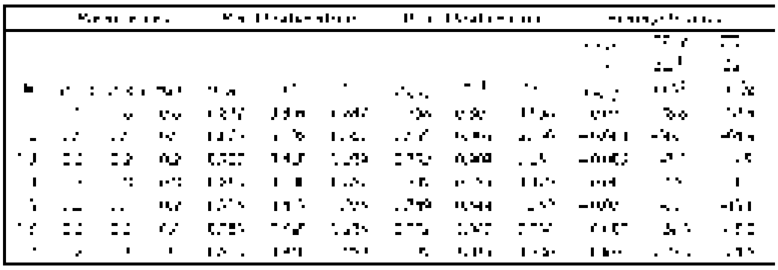

Figure 1: Group competence and imbalance indices for different parameters that are source- and person-independent.

$p_{s\to i}$ varies, but is source- and person-independent. The sharing and absorbing probabilities, $p_{s,i\to}$ and $p_{s,i\leftarrow}$, are also source- and person-independent, to ensure balanced deliberation.

We observe that deliberation often reduces spread imbalance and interpersonal imbalance. In other words, it often reduces Failure 1 and 2, in line with our expectations. Result 1.1, for instance, shows intermediate access, sharing and absorbing. Estimated group competence increases by two percentage points, while the imbalance indices go down. In 1.2, access, sharing and absorbing is high. Consequently, the group starts with higher competence and lower imbalance indices compared to 1.1. Intense deliberation does not increase group competence much further but, due to intense evidence sharing, reduces the imbalance indices to values close to 0.

Result 1.3, however, demonstrates that deliberation, even though balanced, does not guarantee a reduction of Failure 1, as spread imbalance increases. Further runs (see results in the appendix) show that this increase in spread imbalance is associated with relatively low access and sharing, combined with high absorbing. This can also be observed in results 1.6 and 1.7. Moreover, for some more extreme parameter constellations (see 1.7 in Table 1), the group becomes not just more spread imbalanced, but (worse) less competent. Here, the increased spread imbalance outweighs any potential benefits of deliberation. What explains this surprising outcome? First, sharing is so low that typically at most one source is shared. Second, absorbing is so high that this shared source is absorbed by almost everyone. Third, access is quite low, so that the strong emphasis on one source in deliberation is not crowded out by pre-deliberation evidence. This creates high spread imbalance. By focusing on a single source, deliberation reinforces Failure

1; so much so that group competence is lowered.

This interesting phenomenon deserves a closer look. Figure 2 gives an example of how such a post-deliberation source constellation can lead to an incorrect col-

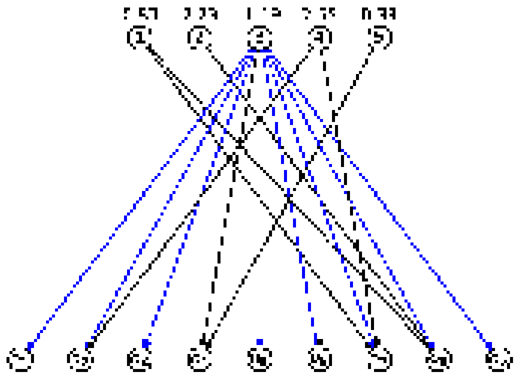

Figure 2: An epistemically harmful outcome of balanced deliberation

lective outcome. The spread of source 3 has risen dramatically after deliberation (blue arrows) compared to before deliberation (black arrows). The figure also shows the values (e_s) above the source nodes. The state here is $x = 1$, which means that source 3 (at $e_3 = -1.19$) emits misleading evidence, pointing in the wrong direction. Unfortunately, the wide spread of that source causes a majority to vote for -1 and the group fails to properly use its evidence (which, in total, strongly points to state 1). In this instance, Failure 1 dominates after deliberation.

While initially surprising, on reflection it is understandable that balanced deliberation reduces group competence in rare cases: A deliberation process in which only little evidence is revealed but then dramatically amplified can be epistemically harmful due to Failure 1, especially when individuals have low access to evidence prior to deliberation.

Interestingly, no such negative effect occurs for interpersonal imbalance. Why not? While perfectly balanced deliberation can create an intense focus on a few sources (as only a few sources might be shared), it cannot create an intense focus on a few individuals. This can only happen for imbalanced deliberation with person-dependent sharing and/or absorbing probabilities. Consequently, Failure 2, as measured as $\mathcal{SI}$, is robustly reduced by balanced deliberation.

4.5 Can balanced deliberation mitigate asymmetric access?

Once parameters can be source- or person-dependent, the parameter space grows quickly. A comprehensive scan of the parameter space is beyond the scope of this paper. Instead we discuss a small selection of instructive scenarios. The current subsection focuses on asymmetric access parameters, while keeping the deliberation parameters balanced.

Can balanced deliberation help to mitigate the effects of asymmetric access? We look at a setting of source- and person-dependent access and display the effects of balanced deliberation in Table 3. In our baseline scenario, the access probability

	Parameters			**Pre-Deliberation**			**Post-Deliberation**			**change from...**		
#	$p_{s\to i}$	$p_{s,i\to}$	$p_{s,i\leftarrow}$	p_{maj}	$\mathcal{SI}$	$\mathcal{II}$	p^+_{maj}	$\mathcal{SI}^+$	$\mathcal{II}^+$	p_{maj} to p^+_{maj}	$\mathcal{SI}$ to $\mathcal{SI}^+$ in %	$\mathcal{II}$ to $\mathcal{II}^+$ in %
	Baseline											
2.0	0.4	0.5	0.5	0.819	0.491	0.964	0.850	0.426	0.663	+0.032	-13.2	-31.2
	Source-dependence (1:4)											
2.1	0.8|0.3	0.5	0.5	0.791	0.760	0.854	0.839	0.612	0.640	+0.048	-19.5	-25.0
	Person-dependence (1:8)											
2.2	0.8|0.35	0.5	0.5	0.820	0.465	1.023	0.851	0.407	0.673	+0.031	-12.5	-34.2
	Person-dependence (5:4)											
2.3	0.6|0.15	0.5	0.5	0.823	0.431	1.175	0.851	0.389	0.692	+0.028	-9.7	-41.1

Figure 3: Balanced deliberation after source- or person-dependent access

is still source- and person-independent, namely at 0.4. In the scenarios to follow, we "redistribute" the probabilities: first, we increase access to one source to 0.8 and reduce access for the 4 other sources to 0.3 (result 2.1). Second, we increase access for one individual to 0.8 and reduce access for the 8 others to 0.35 (result 2.2). Third, we increase access for 5 individuals to 0.6 and reduce access for the 4 others to 0.15 (result 2.3). In all scenarios, this is followed by balanced deliberation with sharing and absorbing probability of 0.5. In the baseline scenario, deliberation reduces both imbalance indices, that is, curtails Failures 1 and 2. But when access is asymmetric, the mitigating effects of deliberation come into view: In the source-dependence case, balanced deliberation is particularly good at reducing Failure 1, measured by spread imbalance. In the two person-dependence cases, balanced deliberation is particularly good at reducing Failure 2, measured as interpersonal imbalance.

4.6 Imbalanced deliberation

There are several reasons why deliberation may be imbalanced. We can distinguish reasons for source- and person-dependence. Source-dependence may be caused, for example, by the fact that some sources are hard to process, lowering the sharing and absorbing probability for that source. Some sources may be unpopular to voice, lowering the sharing probability. Similarly, some sources may "fall on deaf ears" because they are hard to accept for the listener or simply too boring to be attended to, lowering the absorbing probability. Some sources may tend to be held (and affect opinions) subconsciously, which lowers the sharing probability since sharing presumably presupposes awareness. By contrast, person-dependence will be related to person-specific properties. Some voters may have a tendency to publicly announce their evidence (higher sharing probability) or have a higher capacity to listen (absorbing probability). Individuals may also be restricted in their expertise, so that different voters are constrained to sharing or absorbing different evidence, implying both person- and source-dependence. Another interesting case could be that prejudices prevent someone from absorbing evidence from certain persons; however, this case cannot be modelled here because our absorbing probabilities are sharer-independent.

For brevity we only consider selective scenarios. We investigate whether the epistemic problems of asymmetric access are exacerbated if deliberation is unbalanced in the same direction as access. Might such "reinforcing" deliberation be harmful? Not quite, but the coordinated imbalance between deliberation and access seems to reduce the efficiency of deliberation, as the three scenarios in Table 4 suggest. The source-dependence scenario models a situation in which most sources

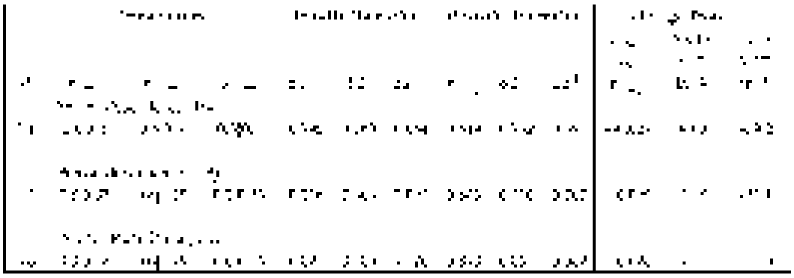

Figure 4: Source- or person-dependence in access, sharing, and absorbing

are hard to access, share and absorb, while one source is easy to access, share, and absorb. Results here are not that clear-cut: Deliberation fails to spread sources more equally and therefore fails to mitigate Failure 1 caused by source-dependent access, but it also does not make it much worse. Compared to the results from balanced deliberation (as in Table 3), however, deliberation is less conducive to good epistemic outcomes. We also look at person-dependence, where either one (result 3.2) or 5 persons (3.3) have a higher access, sharing and absorbing probability than the others. Here, deliberation remains epistemically beneficial and mitigates the significant Failure 2 caused by person-dependent access, but not as much as balanced deliberation did.

5 Opinion structures and deliberation processes generalised

Simple opinion structures exclude irrational opinions (cf. Theorem 1) and are unsuitable for addressing Failure 3 (as will be seen soon), to name just two limitations. Similarly, share-absorb processes are rather special deliberation processes, which for instance exclude biases towards some option, or communication among subgroups. We now widen the perspective by introducing general opinion structures and deliberation processes.

5.1 General opinion structures

Some important real phenomena are beyond simple opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. But they can be accommodated without major departure, by lifting assumptions that were made 'for simplicity'. For instance, real opinions are often affected by *noises*, such as (in a jury's guilty–or-innocent problem) the defendant's skin colour or the room temperature. By allowing the family $(\mathbf{e}_s)$ to consist of general 'influences' which need not all correlate with the state $\mathbf{x}$, some of the $\mathbf{e}_s$ can be (state-independent) noises rather than (state-dependent) evidences. Noises undermine the rationality of opinions obtained for simple structures (Theorem 1). As another example, opinion formation could be a discrete rather than continuous process, which can be modelled using influences $\mathbf{e}_s$ that are discrete rather than Gaussian, e.g., that take only the values 1 ('support for 1') and -1 ('support for -1').

General opinion structures are defined like simple ones, but without the three distributional assumptions (Equiprobable States, Simple Gaussian Evidences, and Independent Sources) and without the assumption that intrapersonal influence aggregation be additive. We write 'g' for the function by which persons aggregate their influences. Simple opinion structures take g to be additive, given by

$$g((e_s)_{s\in S'}) = \sum_{s\in S'} e_s$$

for any influence bundle $(e_s)_{s\in S'}$ across any source set $S' \subseteq S$. Formally, a *(general) opinion structure* is thus a quadruple $(\mathbf{x}, (\mathbf{e}_s)_{s\in S}, (\mathbf{S}_i)_{i\in N}, g) \equiv (\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$ of:

(1) a random variable $\mathbf{x}$, the *state* or *correct option*, taking the value 1 or -1 with arbitrary (non-zero) probabilities;

(2) a family $(\mathbf{e}_s)$, indexed by some set S of *sources* (non-empty and finite), consisting of real-valued random variables, the *influences* from these sources, with arbitrary (discrete or continuous) distributions and arbitrary dependencies with one another and other variables in the opinion structure;

(3) a family $(\mathbf{S}_i)$, indexed by some set $N = \{1, ..., n\}$ of *persons* $(1 \leq n < \infty)$, consisting of random subsets of S, the *source sets* of these persons, again with arbitrary distributions;

(4) a function g, the *influence aggregator*, mapping any influence bundle $(e_s)_{s\in S'}$ $(S' \subseteq S)$ to an 'aggregate influence' $g((e_s)_{s\in S'})$ (technically, a function from $\cup_{S'\subseteq S}\mathbb{R}^{S'}$ to $\mathbb{R}$).

In the default case of an additive influence aggregator g, we denote the structure by '$(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$', taking additivity for granted. Examples are simple opinion structures, in which not only g is additive, but also the three distributional conditions hold.

An influence $\mathbf{e}_s$ is called an *evidence* if $\mathbf{e}_s$ and $\mathbf{x}$ are dependent, and a *noise* otherwise. Simple opinion structures contain only evidences.

As different influences can now be (state-conditionally) dependent, aggregating one's influences additively can be irrational. For instance, *positively* dependent influences, such as evidences from similar sources, are best aggregated subadditively, to avoid multi-counting. Fortunately, g need not be additive – otherwise we would impose irrationality. But g *can* be additive, even when evidences correlate. Indeed, our model is flexible enough to allow opinions to be formed rationally or irrationally, based on simple heuristics or on sophisticated evidence aggregation.

Our entire machinery carries over to a general structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i), g)$. The *opinion* of person i is determined by her (now possibly non-additive) aggregate influence:

$$\mathbf{o}_i = \begin{cases} 1 & \text{if } g((\mathbf{e}_s)_{s \in \mathbf{S}_i}) > 0 \\ -1 & \text{if } g((\mathbf{e}_s)_{s \in \mathbf{S}_i}) < 0 \\ 0 & \text{if } g((\mathbf{e}_s)_{s \in \mathbf{S}_i}) = 0. \end{cases}$$

All other derivative concepts – notably personal competence p_i, the majority opinion $\mathbf{o}_{maj}$, majority competence p_{maj}, and spread imbalance **SI** or $\mathcal{SI}$ – keep their original definitions, except for interpersonal imbalance **II** or $\mathcal{II}$, whose definition should of course be generalised by aggregating personal influences using g rather than additively. Adapting earlier terminology, we now call $(\mathbf{e}_i)_{i \in \mathbf{S}_i}$ person i's *influence bundle* and call $((\mathbf{e}_i)_{i \in \mathbf{S}_i})$ the *influence profile*, since the earlier terms 'evidence bundle' and 'evidence profile' neglect the possibility of non-evidential influences.

5.2 General deliberation processes

Many plausible deliberation processes can be modelled as share-absorb processes, but some cannot. Before presenting a general definition of a 'deliberation process', let us give concrete examples. All examples produce a new source profile $(\mathbf{S}_i^+)$ based on an initial opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. We begin by *deterministic* processes, under which the initial influence profile $((\mathbf{e}_s)_{s \in \mathbf{S}_i})$ fully determines $(\mathbf{S}_i^+)$, followed by *stochastic* processes, under which $((\mathbf{e}_s)_{s \in \mathbf{S}_i})$ determines the probabilities of the possible values of $(\mathbf{S}_i^+)$. Note that a share-absorb process is stochastic if its sharing and absorbing probabilities are neither 0 nor 1, and deterministic if they belong to $\{0, 1\}$.

1. Some deterministic deliberation processes:

1a. *Full sharing.* Under highly idealised circumstances, everyone transmits all his sources to everyone. Here everyone i acquires the same large source set $\mathbf{S}_i^+ = \cup_j \mathbf{S}_j$, hence the same new opinion o_i^+. In a refined variant, everyone transmits all his *evidential* sources to everyone, without sharing sources of noise.

Here, person i's new source set is $\mathbf{S}_i^+ = \mathbf{S}_i \cup (\cup_{j \neq i}\{s \in \mathbf{S}_j : s \text{ is evidential}\})$, containing his initial sources and the 'learnt' evidential sources of others. Indeed, epistemically motivated deliberators will not share noises (which 'pollute' opinions), and perhaps *cannot* share them as they normally affect one subconsciously. Both variants of the process are still share-absorb processes: just set all sharing probabilities and absorbing probabilities to 1, or (for the refined variant) to 1 or 0 depending on whether the source is evidential.

1b. *Topic-dependent sharing.* Everyone shares only sources from her 'area', e.g., her area of expertise or responsibility. Doctors might share only medical evidence, biologists only biological evidence. Technically, each person i has a predefined area $A_i \subseteq S$, and acquires the new source set $\mathbf{S}_i^+ = \mathbf{S}_i \cup (\cup_{j \neq i}(A_j \cap \mathbf{S}_j))$ containing her initial sources and the 'learnt' sources from the areas of others. In a dual variant of the process, everyone *absorbs* (rather than shares) sources from her predefined area, perhaps because of a 'biased' or 'predisposed' perception or attention. Here, person i's new source set is $\mathbf{S}_i^+ = \mathbf{S}_i \cup (A_i \cap (\cup_{j \neq i}\mathbf{S}_j))$. In both variants the process is again still a share-absorb process; the sharing probabilities $p_{s,i\rightarrow}$ are 1 or 0 depending on whether $s \in A_i$ and the absorbing probabilities $p_{s,i\leftarrow}$ are 1, or vice versa for the dual variant.

1c. *Sharing influential sources.* Everyone shares only her most influential source(s). So, any person i acquires the new source set $\mathbf{S}_i^+ = \mathbf{S}_i \cup \big(\cup_{j \neq i}\{s \in \mathbf{S}_j : |\mathbf{e}_s| = \max_{s' \in \mathbf{S}_j} |\mathbf{e}_{s'}|\}\big)$, containing i's initial sources and anyone else's most influential source(s). In a variant of the process, everyone shares her source(s) whose absolute influence exceeds some fixed threshold $\delta \geq 0$, so that $\mathbf{S}_i^+ = \mathbf{S}_i \cup (\cup_{j \neq i}\{s \in \mathbf{S}_j : |\mathbf{e}_s| > \delta\})$, containing i's old sources and anyone else's 'sufficiently influential' sources. In yet other variants, people share their most (or sufficiently) influential *evidential* sources. In no variant is this process a share-absorb process, because sharing is sensitive to influences, not just their sources.

1d. *Biased deliberation.* The group falls into two camps: the set N_1 of supporters of option 1 and the set N_{-1} of supporters of option -1. Supporters of 1 only share sources influencing towards 1. Supporters of -1 only share sources influencing towards -1. Presumably, people are guided by non-epistemic motivations. Under this 'biased' deliberation process, a person i's new source set is $\mathbf{S}_i^+ = \mathbf{S}_i \cup \big(\cup_{j \in N_1 \setminus \{i\}}\{s \in \mathbf{S}_j : \mathbf{e}_s > 0\}\big) \cup \big(\cup_{j \in N_{-1} \setminus \{i\}}\{s \in \mathbf{S}_j : \mathbf{e}_s < 0\}\big)$, containing i's initial sources and biased subsets of anyone else's source sets. Neither this process, nor its dual variant in which absorbing rather than sharing is biased, is a share-absorb process.

2. Some stochastic deliberation processes:

2a. Share-absorb processes with non-extreme sharing and absorbing probabilities are familiar examples, as mentioned.

2b. To obtain a larger class of examples, let us generalise share-absorb processes by allowing the propensity to share or absorb a source to depend on the influence from that source. Technically, for each source s, real number $e \in \mathbb{R}$, and person i, consider two parameters $p_{s,e,i\to}$ and $p_{s,e,i\leftarrow}$ in $[0,1]$, representing the probability that person i shares (respectively absorbs) source s when its influence is e. The parameter family $(p_{s,e,i\to}, p_{s,e,i\leftarrow})_{s\in S, e\in\mathbb{R}, i\in N}$ induces a generalised share-absorb process. It is stochastic if the parameters are strictly between 0 and 1. Interestingly, the *deterministic* deliberation processes in 1c and 1d above are also share-absorb processes of this generalised kind, obtained for certain extreme (i.e., $\{0,1\}$-valued) parameters.

Turning to the general definition, a 'deliberation process' is *some* process that generates a new source profile $(\mathbf{S}_i^+)$. How should we make this 'definition' precise? Observe two systematic features that one would expect the new source profile $(\mathbf{S}_i^+)$ to satisfy, and that are present in all examples above.

First, each new source of anyone was initially owned by someone, as it was learnt from someone. Formally, $\mathbf{S}_i^+ \subseteq \cup_j \mathbf{S}_j$ for all i. Often deliberation is also *monotonic*: no one loses sources during deliberation. Formally, $\mathbf{S}_i \subseteq \mathbf{S}_i^+$ for anyone i. Yet sometimes deliberation 'crowds out' some sources, by making people forget what they knew. So monotonicity will not be required by definition of a deliberation process.

Second, the new source profile $(\mathbf{S}_i^+)$ can be affected by the initial source profile $(\mathbf{S}_i)$ (as in share-absorb processes and in 1a, 1b and 2a), but also by the influences from her sources (as in 1c, 1d and 2b). In sum, $(\mathbf{S}_i^+)$ is affected by the initial influence profile $((\mathbf{e}_s)_{s\in S_i})$.

This leads to the following general definition. A *deliberation process* is a (measurable) function Δ that maps each initial influence profile $((e_s)_{s\in S_i})$ (each value of $((\mathbf{e}_s)_{s\in \mathbf{S}_i})$) to a lottery over profiles (S_i^+) of sets $S_i^+ \subseteq \cup_j S_j$ (technically, a lottery over $(2^{\cup_j S_j})^n$, the set of such source profiles). The probability of an (S_i^+) under this lottery represents how likely this new source profile is under the initial influence profile $((e_s)_{s\in S_i})$. The process Δ is *deterministic* if it always generates a sure lottery (under which some source profile (S_i^+) has probability one), and *stochastic* otherwise. We call a random source profile $(\mathbf{S}_i^+)$ the source profile *generated by* the process Δ if its distribution conditional on the initial opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ is $\Delta(((\mathbf{e}_s)_{s\in\mathbf{S}_i}))$. The generated source profile $(\mathbf{S}_i^+)$ is *essentially unique*, in the sense that its distribution, and more generally the distribution of the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$, is unique.[8]

This general definition unifies the examples 1a–1d and 2a–2b and many more concrete deliberation processes. And it completes the formal framework underlying

[8] That is, if $(\mathbf{S}_i^+)$ and $(\widehat{\mathbf{S}}_i^+)$ are each generated by Δ, then $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ and $(\mathbf{x}, (\mathbf{e}_s), (\widehat{\mathbf{S}}_i^+))$ have the same (joint) distribution.

the following conjecture about how deliberation affects Failure 3.

6 Concluding conjectures about Failure 3

TO BE ADDED

7 References [VERY INCOMPLETE]

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A Rationality of opinions: proof

We now prove Theorem 1, but generalised to *almost simple* opinion structures $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$. The latter are defined exactly like simple opinion structures except that Independent Sources is weakened to the following condition: the source-access events '$s \in \mathbf{S}_i$' (where $s \in S$ and $i \in N$) are independent across sources s and are jointly independent of the state and the evidences, i.e., of $(\mathbf{x}, (\mathbf{e}_s))$. This condition weakens Independent Sources in that the source-access events may now be dependent across persons. The generalisation ensures that the theorem also captures post-deliberation opinions. Indeed, a share-absorb process transforms a *simple* opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ into an *almost simple* one $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$ – 'almost' because deliberation has created interpersonal source dependencies.

We begin by proving a lemma that highlights an astonishing feature of Gaussian distributions.

Lemma 1 *If an opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ satisfies Simple Gaussian Evidences (e.g., is almost simple), then any evidence is proportional to its own log-likelihood-ratio, more precisely*

$$\mathbf{e}_s = \frac{\sigma^2}{2} \log \frac{f(\mathbf{e}_s|1)}{f(\mathbf{e}_s|-1)} \text{ for each } s \in S,$$

where $f(\cdot|x)$ denotes the Gaussian density ('likelihood') function of any evidence $\mathbf{e}_s$ ($s \in S$) given state x ($\in \{\pm 1\}$).

Proof. Let $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ satisfy Simple Gaussian Evidences. Let $s \in S$. Conditional on a state x ($\in \{\pm 1\}$), $\mathbf{e}_s$ is normally distributed with mean x and variance σ^2, hence has a Gaussian density function given by

$$f(e|x) = \frac{1}{\sigma(2\pi)^{1/2}} e^{-\frac{1}{2}\left(\frac{e-x}{\sigma}\right)^2} \text{ for } e \in \mathbb{R}.$$

We have $\mathbf{e}_s = \frac{\sigma^2}{2} \log \frac{f(\mathbf{e}_s|1)}{f(\mathbf{e}_s|-1)}$ because, for all values $e \in \mathbb{R}$ of $\mathbf{e}_s$,

$$\begin{aligned}
\log \frac{f(e|1)}{f(e|-1)} &= \log \frac{\exp\left(-\frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2\right)}{\exp\left(-\frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2\right)} = \log\left[\exp\left(\frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2\right)\right] \\
&= \frac{1}{2}\left(\frac{e+1}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{e-1}{\sigma}\right)^2 = \frac{1}{2\sigma^2}\left[(e+1)^2 - (e-1)^2\right] \\
&= \frac{1}{2\sigma^2}\left[\left(e^2+2e+1\right) - \left(e^2-2e+1\right)\right] = \frac{2}{\sigma^2} e. \blacksquare
\end{aligned}$$

Proof of Theorem 1 generalised to almost simplicity. Fix an almost simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, a person i, and a possible opinion of i, i.e., a random variable $\mathbf{o}$ with values $\{1, 0, -1\}$ based on i's information $(\mathbf{e}_s)_{s \in \mathbf{S}_i}$. We must show that $\mathbb{E}(u(\mathbf{o}_i, \mathbf{x})) \geq \mathbb{E}(u(\mathbf{o}, \mathbf{x}))$, where the utility of any opinion-state pair (o, x) in $\{1, 0, -1\} \times \{1, -1\}$ is the correctness level, given by

$$u(o, x) = \begin{cases} 1 & \text{if } o = x \text{ (correct opinion)} \\ 0 & \text{if } o = -x \text{ (false opinion)} \\ \frac{1}{2} & \text{if } o = 0 \text{ (neutral opinion).} \end{cases}$$

By the structure of u, we must show that

$$Pr(\mathbf{o}_i = \mathbf{x}) + \frac{1}{2} Pr(\mathbf{o}_i = 0) \geq Pr(\mathbf{o} = \mathbf{x}) + \frac{1}{2} Pr(\mathbf{o} = 0).$$

It suffices to show that $\mathbb{E}(u(\mathbf{o}_i, \mathbf{x}) | (\mathbf{e}_s)_{s \in \mathbf{S}_i}) \geq \mathbb{E}(u(\mathbf{o}, \mathbf{x}) | (\mathbf{e}_s)_{s \in \mathbf{S}_i})$.[9] Consider any value $(e_s)_{s \in S_i}$ of i's evidence bundle. Let o_i be the value of $\mathbf{o}_i$ and o the values of $\mathbf{o}$ under $(e_s)_{s \in S_i}$. We must show that

$$\mathbb{E}(u(o_i, \mathbf{x}) | (e_s)_{s \in S_i}) \geq \mathbb{E}(u(o, \mathbf{x}) | (e_s)_{s \in S_i}). \tag{3}$$

Before proving this, we show that

$$Pr(\mathbf{x} = 1 | (e_s)_{s \in S_i}) > (<, =) \ \frac{1}{2} \Leftrightarrow \sum_{s \in S_i} e_s > (<, =) \ 0. \tag{4}$$

[9] Strictly speaking, we must show that this inequality holds *for some versions* of the conditional expectations on both sides. This qualification is necessary because conditional expectations are random variables that are essentially unique rather than unique, i.e., any two versions of a conditional expectation coincide outside a zero-probability event.

Let us only prove the equivalence for '>', as the equivalences for '<' and '=' are analogous. Writing $f(\cdot|x)$ for the Gaussian density function of whatever Gaussian variable given state x ($\in \{\pm 1\}$), we have

$$\begin{aligned} Pr(\mathbf{x}=1|(e_s)_{s\in S_i}) > \tfrac{1}{2} \quad &\Leftrightarrow \quad \tfrac{f((e_s)_{s\in S_i}|1)}{f((e_s)_{s\in S_i}|-1)} > 1 \\ &\Leftrightarrow \quad \textstyle\prod_{s\in S_i} \tfrac{f(e_s|1)}{f(e_s|-1)} > 1 \\ &\Leftrightarrow \quad \textstyle\sum_{s\in S_i} \log \tfrac{f(e_s|1)}{f(e_s|-1)} > 0 \\ &\Leftrightarrow \quad \textstyle\sum_{s\in S_i} e_s > 0. \end{aligned}$$

Here, the first equivalence follows easily from Bayes's rule, using that $Pr(\mathbf{x} = 1) = Pr(\mathbf{x} = -1)$ and also that i's source set is independent of the state and evidences.[10] The second equivalence holds by state-conditional independence of the evidences. The third equivalence holds by applying the logarithm on both sides of the previous inequality. And the fourth equivalence holds by Lemma 1.

We can now derive (3), by proceeding case by case.

Case 1: $o_i = 1$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$, where the inequality holds by (4) as $\sum_{s\in S_i} e_s > 0$.

Subcase 1.1: $o = 1$. Then (3) holds (with '=') because $o_i = o$.

Subcase 1.2: $o = -1$. Then (3) holds (with '>') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = 1 - Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) < \frac{1}{2}$, where the last inequality holds as $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$.

Subcase 1.3: $o = 0$. Then (3) holds (with '>') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = \frac{1}{2}$.

Case 2: $o_i = -1$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = 1 - Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) > \frac{1}{2}$, where the inequality holds because $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) < \frac{1}{2}$ by (4) as $\sum_{s\in S_i} e_s < 0$. An argument similar to that in Case 1 then implies (3).

Case 3: $o_i = 0$. Then $\mathbb{E}(u(o_i, \mathbf{x})|(e_s)_{s\in S_i}) = \frac{1}{2}$. Further, $Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = \frac{1}{2}$, by (4) as $\sum_{s\in S_i} e_s = 0$.

Subcase 3.1: $o = 0$. Then (3) holds (with '=') because $o_i = o$.

Subcase 3.2: $o = 1$. Then (3) holds (with '=') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = 1|(e_s)_{s\in S_i}) = \frac{1}{2}$.

Subcase 3.3: $o = -1$. Then (3) holds (with '=') because $\mathbb{E}(u(o, \mathbf{x})|(e_s)_{s\in S_i}) = Pr(\mathbf{x} = -1|(e_s)_{s\in S_i}) = \frac{1}{2}$. ■

[10] How is this independence condition used here? Informally, this ensures that the nature of the source set S_i is of no extra information, i.e., that only the evidences from those sources carry information. This explains why the likelihood-ratio features only likelihoods of (Gaussian) evidences, not combined likelihoods of those evidences and the source set S_i. To be slightly more explicit, conditionalising on the evidence bundle $(e_s)_{s\in S_i}$ is equivalent to conditionalising first on the source set S_i and then on the evidences from these sources; which however reduces to conditionalising only on the evidences, by the independence condition.

B The analytics of share-absorb processes

The definition of share-absorb processes has been stated informally. The formalisation is obvious. In short, given an opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$, the share-absorb process with parameters $(p_{s,i\rightarrow}, p_{s,i\leftarrow})_{s\in S, i\in N}$ assumes that there exist events 'i shares s' and 'i absorbs s' for any person $i \in N$ and source $s \in S$; that the new source set of any i is $\mathbf{S}_i^+ = \mathbf{S}_i \cup \{s \in S : \text{'}i \text{ absorbs } s\text{'}\}$; that, for any i and s, the probability of 'i shares s' given any initial source profile (S_j) is $p_{s,i\rightarrow}$ if $s \in S_i$ and 0 otherwise$_i$; that, for any i and s, the probability of 'i absorbs s' given any initial source profile (S_j) and any sharing profile is $p_{s,i\leftarrow}$ if [$s \notin S_i$ and someone shares s in the sharing profile] and 0 otherwise (where a 'sharing profile' is a combination of truth values of the sharing events across persons and sources); and, finally, that the access, sharing, and absorbing events are jointly independent of the state and evidences.

How is the new source profile $(\mathbf{S}_i^+)$ distributed? And how is it distributed conditional on the initial evidence profile? We now answer both questions. The first answer completes the description of the new opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i^+))$, as we already know how $(\mathbf{x}, (\mathbf{e}_s))$ is distributed and that $(\mathbf{S}_i^+)$ is independent of $(\mathbf{x}, (\mathbf{e}_s))$. The second answer implies an alternative (and equivalent) definition of the share-absorb process as an abstract deliberation process Δ which maps initial evidence bundles to lotteries over new source profiles.

Fix a simple[11] opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ and a share-absorb process with sharing and absorbing probabilities $(p_{s,i\rightarrow}, p_{s,i\leftarrow})_{s\in S, i\in N}$, generating a new source profile $(\mathbf{S}_i^+)$. The probability of any new source profile (S_i^+) (a value of $(\mathbf{S}_i^+)$) is

$$Pr((S_i^+)) = \prod_{s\in S} \pi_s \tag{5}$$

where, for each source $s \in S$, π_s is the probability that the new set of owners of s is $I_s = \{i : s \in S_i^+\}$, and equals

$$\begin{aligned}\pi_s = &\left(\prod_{i\in I_s} p_{s\rightarrow i}\right)\left(\prod_{i\in \overline{I_s}} \overline{p_{s\rightarrow i}}\right)\left(\prod_{i\in I_s} \overline{p_{s,i\rightarrow}}\right)\\ &+\left(\prod_{i\in \overline{I_s}} \overline{p_{s,i\leftarrow}}\right)\sum_{\varnothing\neq I\subseteq I_s}\left(\prod_{i\in I} p_{s\rightarrow i}\right)\left(\prod_{i\in \overline{I}} \overline{p_{s\rightarrow i}}\right)\left(\overline{\prod_{i\in I} \overline{p_{s,i\rightarrow}}}\right)\left(\prod_{i\in I_s\setminus I} p_{s,i\leftarrow}\right).\end{aligned}$$

Further, the conditional probability of any new source profile (S_i^+) given any initial evidence profile $((e_s)_{s\in S_i})$, or given just (S_i), is

$$Pr((S_i^+)|((e_s)_{s\in S_i}) = Pr((S_i^+)|(S_i)) = \prod_{s\in S} \gamma_s \tag{6}$$

[11] Simplicity could be weakened considerably, to Independent Sources.

where, for each source $s \in S$, γ_s is the probability that the new set of owners of source s is $I_s = \{i : s \in S_i^+\}$ given that the initial one is $J_s = \{i : s \in S_i\}$, which equals

$$\gamma_s = \begin{cases} \left(\overline{\prod_{i \in J_s} \overline{p_{s,i\rightarrow}}}\right) \left(\prod_{i \in I_s \setminus J_s} p_{s,i\leftarrow}\right) \left(\prod_{i \in \overline{I_s}} \overline{p_{s,i\leftarrow}}\right) & \text{if } J_s \subsetneq I_s \\ \left(\overline{\prod_{i \in J_s} \overline{p_{s,i\rightarrow}}}\right) \left(\prod_{i \in \overline{J_s}} \overline{p_{s,i\leftarrow}}\right) + \prod_{i \in J_s} \overline{p_{s,i\rightarrow}} & \text{if } J_s = I_s \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Proof of (5). For each $s \in S$, define two random subgroups, the old set of owners $\mathbf{J}_s = \{i : s \in \mathbf{S}_i\}$ and the new set of owners $\mathbf{I}_s = \{i : s \in \mathbf{S}_i^+\}$. Fix any (S_i^+), and define each I_s ($s \in S$) as above. Note that $(\mathbf{S}_i^+)$ takes the value (S_i^+) if and only if $(\mathbf{I}_s)$ $(= (\mathbf{I}_s)_{s \in S})$ takes the value (I_s) $(= (I_s)_{s \in S})$. Hence, $Pr((S_i^+)) = Pr((I_s))$. The sets $\mathbf{I}_s$ are independent across sources s. So, $Pr((I_s)) = \prod_{s \in S} Pr(I_s)$, and thus

$$Pr((S_i^+)) = \prod_{s \in S} Pr(I_s).$$

Now fix a source s. We calculate $Pr(I_s)$ $(= \pi_s)$. We do this under the assumption that all parameters $p_{s \to i}$ and $p_{s,i\rightarrow}$ are strictly between 0 and 1. This is sufficient since the formula generalises to extreme parameter values by a continuity argument.

First assume $I_s = \varnothing$. Note that $\mathbf{I}_s$ takes the value $\varnothing$ if and only if $\mathbf{J}_s$ takes the value $\varnothing$. The probability of the latter is $\prod_i \overline{p_{s \to i}}$. So, $Pr(I_s) = \prod_i \overline{p_{s \to i}}$. This is what had to be proved, since the claimed expression for $Pr(I_s)$ $(= \pi_s)$ indeed reduces to $\prod_i \overline{p_{s \to i}}$ if $I_s = \varnothing$.

From now on assume $I_s \neq \varnothing$. Then it is certain that $\mathbf{J}_s$ takes a value J_s that satisfies $\varnothing \neq J_s \subseteq I_s$, and each such value J_s has non-zero probability (as each $p_{s \to i}$ is strictly between 0 and 1), so can be conditionalised on. By implication,

$$Pr(I_s) = \sum_{\varnothing \subsetneq J_s \subseteq I_s} Pr(J_s) Pr(I_s | J_s). \tag{8}$$

In this expression, the term $Pr(J_s)$ can be written as

$$Pr(J_s) = \left(\prod_{i \in J_s} p_{s \to i} \right) \left(\prod_{i \in \overline{J_s}} \overline{p_{s \to i}} \right).$$

We now calculate $Pr(I_s | J_s)$. Denote by $!_s$ the event that at least someone shares s. Given the (non-empty) event J_s, each of $!_s$ and $\overline{!_s}$ has non-zero probability (as each $p_{s,i\rightarrow}$ is strictly between 0 and 1), so can be conditionalised on. Hence, $Pr(I_s|J_s)$ is writable as $Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s)Pr(I_s|\overline{!_s}, J_s)$, where $Pr(I_s|\overline{!_s}, J_s)$ is 0 if $J_s \neq I_s$ and 1 if $J_s = I_s$. So,

$$Pr(I_s|J_s) = \begin{cases} Pr(!_s|J_s)Pr(I_s|!_s, J_s) & \text{if } J_s \neq I_s \\ Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s) & \text{if } J_s = I_s. \end{cases}$$

Note that if $J_s = I_s$ we have

$$Pr(\overline{I_s}|J_s) = \prod_{i\in I_s} \overline{p_{s,i\to}}.$$

Upon inserting the derived expressions into (8) and rearranging,

$$\begin{aligned} Pr(I_s) = &\left(\prod_{i\in I_s} p_{s\to i}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s\to i}}\right)\left(\prod_{i\in I_s} \overline{p_{s,i\to}}\right) \\ &+ \sum_{\varnothing\subsetneq J_s\subseteq I_s} \left(\prod_{i\in J_s} p_{s\to i}\right)\left(\prod_{i\in\overline{J_s}} \overline{p_{s\to i}}\right) Pr(!_s|J_s)Pr(!_s|!_s, J_s). \end{aligned}$$

In this,

$$Pr(!_s|J_s)Pr(I_s|!_s, J_s) = \left(\overline{\prod_{i\in J_s} \overline{p_{s,i\to}}}\right)\left(\prod_{i\in I_s\setminus J_s} p_{s,i\leftarrow}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right).$$

So, after rearranging and relabelling the index 'J_s' into 'I',

$$\begin{aligned} Pr(I_s) = &\left(\prod_{i\in I_s} p_{s\to i}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s\to i}}\right)\left(\prod_{i\in I_s} \overline{p_{s,i\to}}\right) \\ &+ \left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right) \sum_{\varnothing\neq I\subseteq I_s} \left(\prod_{i\in I} p_{s\to i}\right)\left(\prod_{i\in\overline{I}} \overline{p_{s\to i}}\right)\left(\overline{\prod_{i\in I} \overline{p_{s,i\to}}}\right)\left(\prod_{i\in I_s\setminus I} p_{s,i\leftarrow}\right). \blacksquare \end{aligned}$$

Proof of (6). Fix any initial evidence profile $((e_s)_{s\in S_i})$ and new source profile (S_i^+). Notation is as above. By definition of share-absorb processes, $Pr((S_i^+)|((e_s)_{s\in S_i})) = Pr((S_i^+)|(S_i))$. I_s and J_s are instances of the random variables $\mathbf{I}_s$ and $\mathbf{J}_s$ in the proof of (5). Since the events $(\mathbf{S}_i^+) = (S_i^+)$ and $(\mathbf{I}_s) = (I_s)$ are equivalent, and the events $(\mathbf{S}_i) = (S_i)$ and $(\mathbf{J}_{s'}) = (J_{s'})$ are also equivalent,

$$Pr((S_i^+)|(S_i)) = Pr((I_s)|(J_{s'})) = \prod_{s\in S} Pr(I_s|(J_{s'})) = \prod_{s\in S} \underbrace{Pr(I_s|J_s)}_{\gamma_s},$$

where the second and third equalities hold by construction of share-absorb processes.

Now fix a source $s \in S$. It remains to prove that $Pr(I_s|J_s)$ $(= \gamma_s)$ is given by (7). We do this under the assumption that each $p_{s,i\to}$ is strictly between 0 and 1. (The generalisation to extreme parameters then follows by continuity.)

If $J_s = I_s = \varnothing$, then $Pr(I_s|J_s) = 1$, because if no one initially owns s, then certainly no one shares or absorbs s.

If $J_s = \varnothing$ and $I_s \neq \varnothing$, then $Pr(I_s|J_s) = 0$, because a source that no one owns is never shared, hence never acquired.

If $J_s \not\subseteq I_s$, i.e., if J_s is not a subset of I_s, then $Pr(I_s|J_s) = 0$, because during deliberation no one loses any initially held sources.

Now assume the remaining case that $\varnothing \neq J_s \subseteq I_s$. As in the proof of (5), denote by $!_s$ the event that at least someone shares s. Given (S_i), each of $!_s$ and $\overline{!_s}$ has non-zero probability (because the parameters $p_{s,i\rightarrow}$ are neither 0 nor 1, and in case of $!_s$ also because $J_s \neq \varnothing$). So we can conditionalise on $!_s$ and on $\overline{!_s}$, and write

$$Pr(I_s|J_s) = Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s)Pr(I_s|\overline{!_s}, J_s).$$

Hence, as $Pr(I_s|\overline{!_s}, J_s)$ is 0 if $J_s \neq I_s$ and 1 if $J_s = I_s$,

$$Pr(I_s|J_s) = \begin{cases} Pr(!_s|J_s)Pr(I_s|!_s, J_s) & \text{if } \varnothing \neq J_s \subsetneq I_s \\ Pr(!_s|J_s)Pr(I_s|!_s, J_s) + Pr(\overline{!_s}|J_s) & \text{if } \varnothing \neq J_s = I_s \end{cases}$$

In this,

$$Pr(\overline{!_s}|J_s) = \prod_{i\in J_s} \overline{p_{s,i\rightarrow}}$$
$$Pr(!_s|J_s) = \overline{\prod_{i\in J_s} \overline{p_{s,i\rightarrow}}}$$
$$Pr(I_s|!_s, J_s) = \left(\prod_{i\in I_s\setminus J_s} p_{s,i\leftarrow}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right).$$

Here, $Pr(I_s|!_s, J_s)$ reduces to $\prod_{i\in\overline{J_s}} \overline{p_{s,i\leftarrow}}$ if $J_s = I_s$. In sum, we have shown that

$$Pr(I_s|J_s) = \begin{cases} 1 & \text{if } J_s = I_s = \varnothing \\ 0 & \text{if } \varnothing = J_s \subsetneq I_s \\ 0 & \text{if } J_s \not\subseteq I_s \\ \left(\overline{\prod_{i\in J_s} \overline{p_{s,i\rightarrow}}}\right)\left(\prod_{i\in I_s\setminus J_s} p_{s,i\leftarrow}\right)\left(\prod_{i\in\overline{I_s}} \overline{p_{s,i\leftarrow}}\right) & \text{if } \varnothing \neq J_s \subsetneq I_s \\ \left(\overline{\prod_{i\in J_s} \overline{p_{s,i\rightarrow}}}\right)\left(\prod_{i\in\overline{J_s}} \overline{p_{s,i\leftarrow}}\right) + \prod_{i\in J_s} \overline{p_{s,i\rightarrow}} & \text{if } \varnothing \neq J_s = I_s \end{cases}$$

Of these six cases, the first can be subsumed under the last, as the formula in the last reduces to 1 if $J_s = I_s = \varnothing$; and the second can be subsumed under the fourth, as the formula in the fourth reduces to 0 if $\varnothing = J_s$. This yields formula (7). ■

C The jury theorems: proofs

Proof of equation (2). Under the assumptions, $\mathbf{o}_{IDEAL} = \mathbf{x}$ holds if and only if total evidence $\sum_{s\in S} \mathbf{e}_s$ has the same sign as $\mathbf{x}$. The probability of this event equals the conditional probability that $\sum_{s\in S} \mathbf{e}_s > 0$ given $\mathbf{x} = 1$, by Simple Gaussian Evidences. Given $\mathbf{x} = 1$, $\sum_{s\in S} \mathbf{e}_s$ is the sum of $|S|$ independent Gaussian variables of mean 1 and variance σ^2, hence is itself a Gaussian variable, with mean $|S|$ and variance $|S|\,\sigma^2$. The probability that such a variable is positive equals the probability that a standard-Gaussian variable is below $\frac{\sqrt{|S|}}{\sigma}$, by a simple transformation. ■

Proof of the Pre-deliberation Jury Theorem. Assume a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ for an infinite population $N = \{1, 2, ...\}$. Notation is as usual.

(a) Fix a group size n. We write $\mathbf{o}_{maj}$ for $\mathbf{o}_{maj,n}$. We first show that $p_{maj} \leq p_{IDEAL}$, i.e., that $Pr(\mathbf{o}_{maj} = \mathbf{x}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$. We begin by proving a general claim:

Claim: For every discrete random variable $\mathbf{z}$ that is independent of the state-evidence combination $(\mathbf{x}, (\mathbf{e}_s))$ (e.g., for $\mathbf{z} = (\mathbf{S}_i)$),

$$Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) > \frac{1}{2}$$

except in a zero-probability event (i.e., except if the combination $((\mathbf{e}_s), \mathbf{z})$ falls into a set into which it falls with zero probability).

To show the claim, note first that such a variable $\mathbf{z}$ is independent of the event $\mathbf{o}_{IDEAL} = \mathbf{x}$ conditional on $(\mathbf{e}_s)$, because $\mathbf{o}_{IDEAL}$ is a function of $(\mathbf{e}_s)$. So, $Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z})$ can be replaced by $Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s))$, which by construction of the ideal opinion $\mathbf{o}_{IDEAL}$ indeed exceeds $\frac{1}{2}$, except in the zero-probability event that $\sum_s \mathbf{e}_s = 0$ (i.e., except if $\mathbf{o}_{IDEAL}$ is zero, hence certainly distinct from $\mathbf{x}$). Q.e.d.

Now choose $\mathbf{z} = (\mathbf{S}_i)$. Then

$$Pr(\mathbf{o}_{maj} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) = \begin{cases} Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) & \text{if } \mathbf{o}_{maj} = \mathbf{o}_{IDEAL} \\ 1 - Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) & \text{if } \mathbf{o}_{maj} = -\mathbf{o}_{IDEAL} \\ 0 & \text{if } \mathbf{o}_{maj} = 0. \end{cases} \tag{9}$$

Here, 'if $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$' means 'if $((\mathbf{e}_s), \mathbf{z})$ takes a value such that $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$', which is well-defined because the value of $((\mathbf{e}_s), \mathbf{z})$ determines the values of $\mathbf{o}_{maj}$ and $\mathbf{o}_{IDEAL}$, hence determines whether $\mathbf{o}_{maj} = \mathbf{o}_{IDEAL}$. And similarly for 'when $\mathbf{o}_{maj} = -\mathbf{o}_{IDEAL}$' and for 'when $\mathbf{o}_{maj} = 0$'.

The 'Claim' and (9) jointly imply that, still for $\mathbf{z} = (\mathbf{S}_i)$,

$$Pr(\mathbf{o}_{maj} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x}|(\mathbf{e}_s), \mathbf{z}) \tag{10}$$

with probability one. By taking expectations on both sides (thereby averaging out $(\mathbf{e}_s)$ and $\mathbf{z}$), we obtain $Pr(\mathbf{o}_{maj} = \mathbf{x}) \leq Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., $p_{maj} \leq p_{IDEAL}$.

Finally, assume Imperfect Access. Then with non-zero probability the variable $\mathbf{z} = (\mathbf{S}_i)$ takes a value such that some source is not accessed by anyone, hence not accessed by a majority. This easily implies that with non-zero probability the second or third case in (9) applies. So, in (10) the '$\leq$' is a '$<$' with non-zero probability. Hence, taking the expectation on both sides of (10) now yields $Pr(\mathbf{o}_{maj} = \mathbf{x}) < Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., $p_{maj} < p_{IDEAL}$.

(b) We now show the convergence claim, assuming Access Competence. By this assumption, there is an $\epsilon > 0$ such that $p_{s\to i} \geq 2^{-1/|S|}+\epsilon$ for all s and i. Consider a person i. The probability of having full source set S satisfies $Pr(\mathbf{S}_i = S) \geq \frac{1}{2}+\epsilon^{|S|}$, because

$$Pr(\mathbf{S}_i = S) = \prod_{s\in S} p_{s\to i} \geq \prod_{s\in S}(2^{-1/|S|}+\epsilon) = \left(2^{-1/|S|}+\epsilon\right)^{|S|} \geq (2^{-1/|S|})^{|S|}+\epsilon^{|S|} = \frac{1}{2}+\epsilon^{|S|}.$$

Since the full-access events '$\mathbf{S}_i = S$' ($i = 1, 2, ...$) are mutually independent (by Independent Sources) and each of probability at least $\frac{1}{2}+\epsilon^{|S|}$, the probability that the proportion of members with full access exceeds $\frac{1}{2}$ (the event $\frac{\#\{i\in\{1,...,n\}:\mathbf{S}_i=S\}}{n} > \frac{1}{2}$) tends to one as $n \to \infty$, by the law of large numbers. In other words, the probability of a majority with full access (the event $\#\{i \in \{1, ..., n\} : \mathbf{S}_i = S\} > \frac{n}{2}$) tends to 1 as $n \to \infty$. Meanwhile, full access implies an ideal opinion (i.e., $\mathbf{S}_i = S$ implies $\mathbf{o}_i = \mathbf{o}_{IDEAL}$), and so a majority with full access implies a majority with the ideal opinion (i.e., $\#\{i \in \{1, ..., n\} : \mathbf{S}_i = S\} > \frac{n}{2}$ implies $\mathbf{o}_{maj,n} = \mathbf{o}_{IDEAL}$). So, also the probability of an ideal majority opinion converges to one: $Pr(\mathbf{o}_{maj,n} = \mathbf{o}_{IDEAL}) \to 1$. This implies that $Pr(\mathbf{o}_{maj,n} = \mathbf{x}) \to Pr(\mathbf{o}_{IDEAL} = \mathbf{x})$, i.e., that $p_{maj,n} \to p_{IDEAL}$. ■

Proof of the Post-deliberation Jury Theorem. Assume a simple opinion structure $(\mathbf{x}, (\mathbf{e}_s), (\mathbf{S}_i))$ and a share-absorb process, both for an infinite population $N = \{1, 2, ...\}$. The usual notation applies.

(a) The non-asymptotic claim holds by a version of the proof of part (a) of the Pre-deliberation Jury Theorem. One should substitute $\mathbf{o}^+_{maj}$ for $\mathbf{o}_{maj}$, and apply the 'Claim' with $\mathbf{z} = (\mathbf{S}^+_i)$ rather than $\mathbf{z} = (\mathbf{S}_i)$, which is possible since $(\mathbf{S}^+_i)$ is also independent of $(\mathbf{x}, (\mathbf{e}_s))$.

(b) We now turn to the asymptotic claim. We shall face the difficulty of interpersonal correlations between post-deliberation source sets. The weak law of large numbers in Pivato's (2017) version for correlated variables will ultimately come to help, but first several claims must be established. We assume Acquisition Competence (needed from Claim b5) and Non-vanishing Sharing Competence (needed from Claim b4).[12]

Claim b1: For any source $s \in S$, group size $n \in \{1, 2, ...\}$, and group member $i \in \{1, ..., n\}$, the probability that another member shares s is

$$p_{s,i,n} = \overline{\prod_{j\in\{1,...,n\}\setminus\{i\}} \overline{p_{s\to j}p_{s,j\to}}}. \tag{11}$$

[12] The proof repeatedly invokes the event that a person i shares a source s, or that i absorbs s. Strictly speaking, this presupposes that such sharing or absorbing events are formal events in the probability space (initially, they are 'only' part of the informal story behind the definition of share-absorb processes, as this definition does not officially require $p_{s,i\to}$ and $p_{s,i\leftarrow}$ to be probabilities of formal events). This presupposition is unproblematic: it can be made without loss of generality.

The probability is given by (11) because it equals the probability that it is *not* the case that each other member j does *not* share s, where j shares s with probability $p_{s\to j}p_{s,j\to}$, the product of the probabilities of accessing s and of sharing an accessed s. Q.e.d.

Claim b2: For any $s \in S$, $n \in \{1, 2, ...\}$, member $i \in \{1, ..., n\}$, the probability that some member other than i shares s and then i absorbs s, given that i has not accessed s initially, is $p_{s,i,n}p_{s,i\leftarrow}$.

The claim holds because the relevant probability is the product of the probability that someone else shares s, i.e., $p_{s,i,n}$ (by Claim b1), and the probability that i absorbs a shared source s, i.e., $p_{s,i\leftarrow}$. Q.e.d.

Claim b3: For any $s \in S$, $n \in \{1, 2, ...\}$, and $i \in \{1, ..., n\}$, $Pr(s \in \mathbf{S}^+_{i,n}) = \overline{\overline{p_{s\to i}} \times \overline{p_{s,i,n}p_{s,i\leftarrow}}}$.

This holds because i does *not* hold s post-deliberation if and only if i does not initially access s (probability: $\overline{p_{s\to i}}$) and i does not absorb s (probability: $\overline{p_{s,i,n}p_{s,i\leftarrow}}$). Q.e.d.

Claim b4: For any $s \in S$ and $i \in \{1, 2, ...\}$, $P(s \in \mathbf{S}^+_{i,n}) \to \overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}}$ as $n \to \infty$.

Fix s and i. By Claim b3 it suffices to show that $p_{s,i,n} \to 1$, i.e., that $\prod_{j\in\{1,...,n\}\setminus\{i\}} \overline{p_{s\to j}p_{s,j\to}} \to 0$. By Non-vanishing Sharing Competence, $p_{s\to j}p_{s,j\to} \not\to 0$ as $j \to \infty$, and hence $\overline{p_{s\to j}p_{s,j\to}} \not\to 1$ as $j \to \infty$. In consequence, $\prod_{j\in\{1,...,n\}\setminus\{i\}} \overline{p_{s\to j}p_{s,j\to}} \to 0$ as $n \to \infty$. Q.e.d.

Claim b5: For any $i \in \{1, 2, ...\}$, the full-access probability $Pr(\mathbf{S}^+_{i,n} = S)$ converges to a value of at least $\frac{1}{2} + \epsilon^{|S|}$ as $n \to \infty$, where $\epsilon > 0$ is the threshold in Acquisition Competence (which is independent of i).

Fix a person i. We have $Pr(\mathbf{S}^+_{i,n} = S) = \prod_{s\in S} Pr(s \in \mathbf{S}^+_{i,n})$, because the access events '$s \in \mathbf{S}^+_{i,n}$' are independent across sources s as a consequence of the fact that the pre-deliberation access events '$s \in \mathbf{S}_i$' are independent across sources (by Source Independence) and the share-absorb process operates independently across sources. So, using Claim b4, $Pr(\mathbf{S}^+_{i,n} = S) \to \prod_{s\in S} \overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}}$ as $n \to \infty$. Now choose $\epsilon > 0$ as in Acquisition Competence. Then, for all s, $\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}} \leq 1 - 2^{-1/|S|} - \epsilon$, i.e., $\overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}} \geq 2^{-1/|S|} + \epsilon$. So,

$$\prod_{s\in S} \overline{\overline{p_{s\to i}} \times \overline{p_{s,i\leftarrow}}} \geq \left(2^{-1/|S|} + \epsilon\right)^{|S|} \geq (2^{-1/|S|})^{|S|} + \epsilon^{|S|} = \frac{1}{2} + \epsilon^{|S|}.$$

Hence $\lim_{n\to\infty} Pr(\mathbf{S}^+_{i,n} = S) \geq \frac{1}{2} + \epsilon^{|S|}$. Q.e.d.

Claim b6: For all $n \in \{1, 2, ...\}$ and distinct $i, j \in \{1, ..., n\}$, the covariance between i's and j's full access satisfies

$$Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) \leq \prod_{s\in S}\prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \prod_{s\in S}\prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}}$$

Fix $n \in \{1, 2, ...\}$ and distinct $i, j \in \{1, ..., n\}$. Then

$$
\begin{aligned}
Cov(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) &= Pr(\mathbf{S}^+_{i,n} = S, \mathbf{S}^+_{j,n} = S) - \prod_{k=i,j} Pr(\mathbf{S}^+_{k,n} = S) \\
&= \prod_{s \in S} Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) - \prod_{s \in S} \prod_{k=i,j} Pr(s \in \mathbf{S}^+_{k,n}),
\end{aligned}
$$

where the second equality holds by independence across sources of the access events. Since $Pr(s \in \mathbf{S}^+_{k,n}) = \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}}$ by Claim b3, it suffices to show that

$$
Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) \leq \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} \text{ for all } s \in S.
$$

This holds since, letting E be the event that s is shared by someone in $\{1, ..., n\}$,

$$
\begin{aligned}
Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n}) &\leq Pr(s \in \mathbf{S}^+_{i,n}, s \in \mathbf{S}^+_{j,n} | E) \\
&= \prod_{k=i,j} Pr(s \in \mathbf{S}^+_{k,n} | E) = \prod_{k=i,j} \overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}},
\end{aligned}
$$

where the first equality holds by independence between '$s \in \mathbf{S}^+_{i,n}$' and '$s \in \mathbf{S}^+_{i,n}$' given E, and the second equality holds because s is held ex-post if and only if it is *not* the case that s is *not* accessed ex-ante (probability: $\overline{p_{s\to k}}$) and *not* absorbed ex-post (probability: $\overline{p_{s,i\leftarrow}}$). Q.e.d.

Claim b7: : $\min_{s \in S, k \leq n} p_{s,k,n} \to 1$ as $n \to \infty$.

For all s and n, choose a person $i_{s,n} \in \{1, ..., n\}$ such that $p_{s\to i_{s,n}} p_{s,i_{s,n}\to} = \max_{k \leq n} p_{s\to k} p_{s,k\to}$. By construction,

$$
\min_{k \leq n} p_{s,k,n} = \overline{\prod_{j \in \{1,...,n\} \setminus \{i_{s,k}\}} \overline{p_{s\to j} p_{s,j\to}}}.
$$

By Non-vanishing Sharing Competence, $p_{s\to j} p_{s,j\to} \not\to 0$ as $n \to \infty$. This implies that $\prod_{j \in \{1,...,n\} \setminus \{i_{s,k}\}} \overline{p_{s\to j} p_{s,j\to}} \to 0$. So, $\min_{k \leq n} p_{s,k,n} \to 1$. Hence, as $|S|$ is finite, $\min_{s \in S, k \leq n} p_{s,k,n} \to 1$. Q.e.d.

Claim b8: $\delta_n \equiv \max_{s \in S, k \leq n} \left(\overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}} \right) \to 0$ as $n \to \infty$.

For all $s \in S$, $n \in \{1, 2, ...\}$ and $k \leq n$, we have

$$
\begin{aligned}
\overline{\overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}}} - \overline{\overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}}} &= \overline{p_{s\to k}} \times \overline{p_{s,k,n}p_{s,k\leftarrow}} - \overline{p_{s\to k}} \times \overline{p_{s,k\leftarrow}} \\
&= \overline{p_{s\to k}} \left(\overline{p_{s,k,n}p_{s,k\leftarrow}} - \overline{p_{s,k\leftarrow}} \right) \\
&= \overline{p_{s\to k}} \left(p_{s,k\leftarrow} - p_{s,k,n}p_{s,k\leftarrow} \right) \\
&= \overline{p_{s\to k}} p_{s,k\leftarrow} \left(1 - p_{s,k,n} \right) \\
&\geq 1 - p_{s,k,n}.
\end{aligned}
$$

This lower bound implies the desired convergence via Claim b7. Q.e.d.

Claim b9: The average covariance of full access between group members converges to zero, i.e.,

$$\frac{1}{n^2}\sum_{i,j=1}^{n} Cov(\mathbf{S}_{i,n}^{+}=S,\mathbf{S}_{j,n}^{+}=S)\to 0 \text{ as } n\to\infty.$$

It suffices to show that $Cov(\mathbf{S}_{i,n}^{+}=S,\mathbf{S}_{j,n}^{+}=S)\leq 1$ whenever $i=j$ $(\leq n)$ and that $\max_{i,j\leq n,i\neq i} Cov(\mathbf{S}_{i,n}^{+}=S,\mathbf{S}_{j,n}^{+}=S)\to 0$, by an argument (which uses that each $Cov(\mathbf{S}_{i,n}^{+}=S,\mathbf{S}_{j,n}^{+}=S)$ is positive). The former is obvious. We now show the latter. By Claim b6 and the positivity of all the covariances, it is enough to prove that, for any distinct i,j,

$$\prod_{s\in S}\prod_{k=i,j} a_{s,k}-\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}\to 0 \text{ as } n\to\infty,$$

where $a_{s,k}=\overline{\overline{p_{s\to k}}\times\overline{p_{s,k\leftarrow}}}$ and $a_{s,k,n}=\overline{\overline{p_{s\to k}}\times\overline{p_{s,k,n}p_{s,k\leftarrow}}}$. Fix distinct i,j. Note that $a_{k,s}=(a_{s,k}-a_{s,k,n})+a_{s,k,n}\leq\delta_n+a_{s,k,n}$, by Claim b8. So it suffices to show that

$$\prod_{s\in S}\prod_{k=i,j}(\delta_n+a_{s,k,n})-\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}\to 0 \text{ as } n\to\infty. \tag{12}$$

By multiplying out, check that $\prod_{s\in S}\prod_{k=i,j}(\delta_n+a_{s,k,n})$ is a polynomial in δ_n (of order $2\,|S|$) whose constant term is '$+\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}$'. This constant term cancels out with '$-\prod_{s\in S}\prod_{k=i,j} a_{s,k,n}$', so that the expression in (12) is a polynomial in δ_n with *zero* constant term. As $n\to\infty$, δ_n converges to zero (by Claim b8), and hence so does any polynomial in δ_n with zero constant term. This proves the convergence in (12). Q.e.d.

Claim b10: $p_{maj,n}^{+}\to p_{IDEAL}$. (This completes the proof.)

Since every person i's full-access event $\mathbf{S}_{i,n}^{+}=S$ has probability converging to $\frac{1}{2}+\epsilon^{|S|}$ as $n\to\infty$ by Claim b5, while the average covariance of these events tends to zero by Claim b9, the probability that the proportion of members with full access exceeds $\frac{1}{2}$ (the event $\frac{\#\{i\in\{1,...,n\}:\mathbf{S}_{i,n}^{+}=S\}}{n}>\frac{1}{2}$) tends to one, by the weak law of large numbers in Pivato's (2017) version for correlated variables.[13]

Equivalently, the probability of a majority with full access (the event $\#\{i\in\{1,...,n\}:\mathbf{S}_{i,n}^{+}=S\}>\frac{n}{2}$) tends to 1. Since (a majority with) full access implies (a majority with) an ideal opinion, also the probability of an ideal majority opinion converges to 1: $Pr(\mathbf{o}_{maj,n}^{+}=\mathbf{o}_{IDEAL})\to 1$. So, $Pr(\mathbf{o}_{maj,n}^{+}=\mathbf{x})\to Pr(\mathbf{o}_{IDEAL}=\mathbf{x})$, i.e., $p_{maj,n}^{+}\to p_{IDEAL}$. ■

D More simulation results

TO BE ADDED

[13] This version of the law follows from the proof of Pivato's Theorem 5.2 (i.e., from Claim 2 in that proof, combined with Chebyshev's Inequality). A closely related result is Proposition A2 in Pivato (2016).