Opinion pooling on general agendas: linearity or just neutrality?

An appendix to "Opinion pooling on general agendas" (F. Dietrich and C. List)[1]

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In "Opinion pooling on general agendas", we characterize linear and neutral opinion pooling functions $F : \mathcal{P}^n \to \mathcal{P}$, where unlike in the classical opinion pooling problem the agenda X of relevant events need not form a σ-algebra. These characterizations are based on two conditions on the pooling function: independence and implication-preservation. The latter condition is stronger than the standard (Pareto-like) condition of zero-preservation. In the present appendix, (i) we show that our characterizations would not in general hold if instead of implication-preservation we merely require zero-preservation; but (ii) for an interesting class of agendas, zero-preservation (still together with independence) suffices to force pooling to be neutral (Theorem 3) while leaving room for non-linear pooling (Theorem 4). These results suggest that without invoking the requirement of implication-preservation a normative defense of linear pooling becomes difficult. The framework and notation is the same as in the original paper.[2]

Say that a relevant event $A \in X$ *conditionally entails* another one $B \in X$ (written $A \vdash^* B$) if $\{A\} \cup Y$ entails B (i.e. $\cap_{C \in \{A\} \cup Y} C \subseteq B$) for some countable set $Y \subseteq X$ that is consistent with A (i.e. $\cap_{C \in \{A\} \cup Y} C \neq \emptyset$) and with B^c (i.e. $\cap_{C \in \{B^c\} \cup Y} C \neq \emptyset$). The agenda X is *pathconnected* if for any two events $A, B \in X \backslash \{\emptyset, \Omega\}$ there exist events $A_1, ..., A_k \in X$ ($k \geq 1$) such that $A = A_1 \vdash^* A_2 \vdash^* ... \vdash^* A_k = B$. In other words, any two contingent events in the agenda can be connected by a path of conditional entailments.[3] For instance, $X := \{A, A^c : A \subseteq \mathbf{R}$ is a bounded interval$\}$ is a pathconnected agenda (a subset of the Borel-σ-algebra Σ over $\Omega = \mathbf{R}$).[4] One easily shows that

[1] The first version of the main paper still contained this appendix.

[2] Note however that, as zero-preservation does not anymore refer to the probability of events outside the agenda X, one might for the purpose of the present appendix re-define a pooling function as a mapping from $\mathcal{P}_X^n$ to $\mathcal{P}_X$ rather than for $\mathcal{P}^n$ to $\mathcal{P}$, where $\mathcal{P}_X$ is the set of functions $P : X \to [0, 1]$ that can be extended to a probability measure on Σ.

[3] Conditional entailment and pathconnecdness are closely related to the notions of conditional entailment and total blockedness introduced in a binary (not probabilistic) setup by Nehring and Puppe, "Strategy-proof social choice on single-peaked domains: possibility, impossibility and the space between", working paper (2002). The strategy to prove Theorem 3 reminds of arguments made in that paper (in particular, our lemma has an analogue in the binary setup).

[4] For example, a path of conditional entailments between the intervals $[0, 1]$ and $[2, 3]$ can

pathconnected agendas are non-simple; but many non-simple agendas are not pathconnected.

We now give a characterization of neutral pooling based on requiring just zero-preservation, not implication-prerservation.

Theorem 3 (a) *For a pathconnected agenda, every independent zero-preserving pooling function is neutral.*

(b) *For a non-pathconnected finite agenda, not every independent zero-preserving pooling function is neutral.*

So, for a pathconnected agenda, independence leads to neutrality. Does it even lead to linearity? The answer is negative, as the next theorem shows.

Theorem 4 *For some pathconnected agenda X (in some σ-algebra Σ over some set of worlds Ω), not every neutral zero-preserving pooling function is linear.*

The following lemma is central for proving part (a) of Theorem 3.

Lemma *For any independent and zero-preserving pooling function, $A \vdash^* B$ implies $D_A \leq D_B$ for all relevant events $A, B \in X$ (where $D_A : [0,1]^n \to [0,1]$ is the local pooling criterion for A, and D_B is that for B).*

Proof. Let F, A, B, D_A, D_B be as specified, and assume $A \vdash^* B$, say in virtue of the set $Y \subseteq X$. Let $x = (x_1, ..., x_n) \in [0,1]^n$. We show that $D_A(x) \leq D_B(x)$. As $\cap_{C \in \{A\} \cup Y} C$ has empty intersection with B^c (by the conditional entailment), it equals its intersection with B; in particular, $\cap_{C \in \{A,B\} \cup Y} C \neq \emptyset$. Similarly, as $\cap_{C \in \{B^c\} \cup Y} C$ has empty intersection with A, it equals its intersection with A^c; in particular, $\cap_{C \in \{A^c, B^c\} \cup Y} C \neq \emptyset$. Hence there are worlds $\omega \in \cap_{C \in \{A,B\} \cup Y} C$ and $\omega' \in \cap_{C \in \{A^c, B^c\} \cup Y} C$. For each individual i, consider the probability measure $P_i : \Sigma \to [0,1]$ defined by

$$P_i := x_i \delta_\omega + (1 - x_i)\delta_{\omega'},$$

where $\delta_\omega, \delta_{\omega'} : \Sigma \to [0,1]$ denote the Dirac-measures in ω and ω', respectively. As each P_i satisfies $P_i(A) = P_i(B) = x_i$, we have

$$\begin{aligned} P_{P_1,...,P_n}(A) &= D_A(P_1(A), ..., P_n(A)) = D_A(x), \\ P_{P_1,...,P_n}(B) &= D_B(P_1(B), ..., P_n(B)) = D_B(x). \end{aligned}$$

Further, for each P_i and each $C \in Y$ we have $P_i(C) = 1$, so that $P_{P_1,...,P_n}(C) = 1$ (by zero-preservation), and hence $P_{P_1,...,P_n}(\cap_{C \in Y} C) = 1$ since the intersection of

be constructed as follows: $[0,1] \vdash^* [0,3]$ (one may conditionalise on the empty set of events $Y = \emptyset$, i.e. the entailment is *un*conditional), and $[0,3] \vdash^* [2,3]$ (one may conditionalise on $Y = \{[2,4]\}$.

countably many events of probability one has again probability one. So

$$\begin{aligned} P_{P_1,...,P_n}(\cap_{C\in\{A\}\cup Y}C) &= P_{P_1,...,P_n}(A) = D_A(x), \\ P_{P_1,...,P_n}(\cap_{C\in\{B\}\cup Y}C) &= P_{P_1,...,P_n}(B) = D_B(x). \end{aligned}$$

Now $P_{P_1,...,P_n}(\cap_{C\in\{A\}\cup Y}C) \leq P_{P_1,...,P_n}(\cap_{C\in\{B\}\cup Y}C)$ since

$$\cap_{C\in\{A\}\cup Y}C = \cap_{C\in\{A,B\}\cup Y} \subseteq \cap_{C\in\{B\}\cup Y}C$$

(for the equality, see an earlier argument). So $D_A(x) \leq D_B(x)$, as desired. ■

Proof of Theorem 3. (a) Let X be pathconnected and F independent and zero-preserving. If $X = \{\emptyset, \Omega\}$, F is obviously neutral, as desired. Now let $X \neq \{\emptyset, \Omega\}$ and write D_A for the local pooling criterion of any contingent event $A \in X\backslash\{\emptyset, \Omega\}$. As X is pathconnected, repeated application of the above lemma yields $D_A \leq D_B$ for all $A, B \in X\backslash\{\emptyset, \Omega\}$, and hence $D_A = D_B$ for all $A, B \in X\backslash\{\emptyset, \Omega\}$. Define D as the common pooling criterion D_A of all $A \in X\backslash\{\emptyset, \Omega\}$. We complete the neutrality proof by showing that D also works as a pooling criterion for $\emptyset$ and Ω. Consider any $P_1, ..., P_n \in \mathcal{P}$. By definition of probability measures,

$$\begin{aligned} P_1(\emptyset) &= ... = P_n(\emptyset) = P_{P_1,...,P_n}(\emptyset) = 0, \\ P_1(\Omega) &= ... = P_n(\Omega) = P_{P_1,...,P_n}(\Omega) = 1. \end{aligned}$$

So it suffices to show that $D(0, ..., 0) = 0$ and $D(1, ..., 1) = 1$, which follows from zero-preservation.

(b) Now let X be finite and not pathconnected. By an argument in the main paper, we may assume that the σ-algebra generated by X is the entire σ-algebra Σ. Notationally, for any sub-σ-algebra $\bar{\Sigma} \subseteq \Sigma$, let $\mathcal{A}(\bar{\Sigma})$ be its set of atoms (i.e. with respect to set-inclusion minimal non-empty elements). We now define a pooling function and show that it has the desired properties. As an ingredient to the definition, let $D' : [0,1]^n \to [0,1]$ and $D'' : [0,1]^n \to [0,1]$ be the local decision rules of two distinct linear pooling functions; and let $\bar{A} \in X\backslash\{\emptyset, \Omega\}$ be a (by assumption existing) event such that not for all $A \in X\backslash\{\emptyset, \Omega\}$ there is $\bar{A} \vdash\vdash^* A$, where "$\vdash\vdash^*$" stands for the existence of a finite path of conditional entailments as in the definition of pathconnectedness. Consider any profile $(P_1, ..., P_n) \in \mathcal{P}^n$. To define a probability measure $P_{P_1,...,P_n} : \Sigma \to [0,1]$, we start by defining probability measures on two sub-σ-algebras of Σ, denoted Σ' and Σ'' and defined as the σ-algebras generated by the sets

$$\begin{aligned} X' &:= \{A \in X : \bar{A} \vdash\vdash^* B \text{ for both } B \in \{A, A^c\}\}, \\ X'' &:= \{A \in X : \bar{A} \vdash\vdash^* B \text{ for no } B \in \{A, A^c\}\}, \end{aligned}$$

respectively. Let $P'_{P_1,...,P_n} : \Sigma' \to [0,1]$ and $P''_{P_1,...,P_n} : \Sigma'' \to [0,1]$ be defined by

$$\begin{aligned} P'_{P_1,...,P_n}(A) &= D'(P_1(A), ..., P_n(A)) \text{ for all } A \in \Sigma', \\ P''_{P_1,...,P_n}(A) &= D''(P_1(A), ..., P_n(A)) \text{ for all } A \in \Sigma''. \end{aligned}$$

The functions are indeed probability measures (on Σ' resp. Σ''), as they are linear averages of of probability measures.

Claim 3. The σ-algebras Σ' and Σ'' are logically independent, that is: if $A' \in \Sigma'$ and $A'' \in \Sigma''$ are non-empty, so is $A' \cap A''$.

Suppose the contrary. Then, as each non-empty element of Σ' is a superset of an atom of Σ' and hence of a non-empty intersection of events in X', and similarly for Σ'', there are consistent sets $Y' \subseteq X'$ and $Y'' \subseteq X''$ such that $Y' \cup Y''$ is inconsistent. Let Y be a minimal inconsistent subset of $Y' \cup Y''$. Y is not a subset of any of Y' and Y'', because the latter sets are consistent. So there are $A \in Y \cap X'$ and $B \in Y \cap X''$. Note that $A \vdash^* B^c$, a contradiction since $A \in X'$ and $B^c \in X''$, q.e.d.

We now extend the measures $P'_{P_1,...,P_n}$ and $P''_{P_1,...,P_n}$ to a probability measure on the σ-algebra $\tilde{\Sigma}$ generated by $\Sigma' \cup \Sigma''$, i.e. generated by $X' \cup X''$, in such a way that the events in Σ' are probabilistically independent of those in Σ''. By Claim 3, the atoms of $\tilde{\Sigma}$ are precisely the intersections of an atom of Σ' and one of Σ'': $\mathcal{A}(\tilde{\Sigma}) = \{A' \cap A'' : A' \in \mathcal{A}(\Sigma'), A'' \in \mathcal{A}(\Sigma'')\}$. Let $\tilde{P}_{P_1,...,P_n}$ be the unique measure on $\tilde{\Sigma}$ that behaves as follows on the atoms:

$$\tilde{P}_{P_1,...,P_n}(A' \cap A'') = P'_{P_1,...,P_n}(A') P''_{P_1,...,P_n}(A'') \tag{1}$$

for all $A' \in \mathcal{A}(\Sigma')$ and all $A'' \in \mathcal{A}(\Sigma'')$. This measure is indeed a probability measure, because

$$\begin{aligned}
\sum_{A \in \mathcal{A}(\tilde{\Sigma})} \tilde{P}_{P_1,...,P_n}(A) &= \sum_{A' \in \mathcal{A}(\Sigma'), A'' \in \mathcal{A}(\Sigma'')} P'_{P_1,...,P_n}(A') P''_{P_1,...,P_n}(A'') \\
&= \sum_{A' \in \mathcal{A}(\Sigma')} P'_{P_1,...,P_n}(A') \underbrace{\sum_{A'' \in \mathcal{A}(\Sigma'')} P''_{P_1,...,P_n}(A'')}_{=1} \\
&= 1.
\end{aligned}$$

As one easily checks, restricting $\tilde{P}_{P_1,...,P_n}$ to Σ' resp. Σ'' gives $P'_{P_1,...,P_n}$ resp. $P''_{P_1,...,P_n}$, and so

$$\tilde{P}_{P_1,...,P_n}(A) = \begin{cases} D'(P_1(A), ..., P_n(A)) & \text{for all } A \in \Sigma' \\ D''(P_1(A), ..., P_n(A)) & \text{for all } A \in \Sigma''. \end{cases} \tag{2}$$

Before we can extend $\tilde{P}_{P_1,...,P_n}$ to the full σ-algebra Σ, we first prove another claim. For all $A \in X$ such that $\bar{A} \vdash\vdash^* A$ but not $\bar{A} \vdash\vdash^* A^c$, define

$$A_{P_1,...,P_n} := \begin{cases} A & \text{if } P_i(A) > 0 \text{ for some } i \\ A^c & \text{if } P_i(A) = 0 \text{ for all } i. \end{cases}$$

Claim 4. For all atoms C of $\tilde{\Sigma}$ $(= \sigma(X' \cup X''))$ with $\tilde{P}_{P_1,...,P_n}(C) > 0$, the event $C \cap (\cap_{A \in X : \bar{A} \vdash\vdash^* A \text{ and not } \bar{A} \vdash\vdash^* A^c} A_{P_1,...,P_n})$ is an atom of Σ.

Let C be as specified, and write $C_{P_1,\dots,P_n}$ for the event in question. As noted above, C takes the form $C = A' \cap A''$ with $A' \in \mathcal{A}(\Sigma')$ and $A'' \in \mathcal{A}(\Sigma'')$. By $P(C) > 0$ and (1), we have $\tilde{P}_{P_1,\dots,P_n}(A') > 0$ and $\tilde{P}_{P_1,\dots,P_n}(A'') > 0$. As $A' \in \mathcal{A}(\Sigma')$, we may write $A' = \cap_{A \in Y'} A$ for some set $Y' \subseteq X'$ containing exactly one member of each pair $A, A^c \in X'$. Similarly, $A'' = \cap_{A \in Y''} A$ for some set $Y'' \subseteq X''$ containing exactly one member of each pair $A, A^c \in X''$. Note also that $\cap_{A \in X: \bar{A} \vdash\vdash^* A \text{ and not } \bar{A} \vdash\vdash^* A^c} A_{P_1,\dots,P_n}$ can be written as $\cap_{A \in Y_{P_1,\dots,P_n}} A$, where the set

$$Y_{P_1,\dots,P_n} = \{A_{P_1,\dots,P_n} : A \in X,\ \bar{A} \vdash\vdash^* A,\ \text{not } \bar{A} \vdash\vdash^* A^c\}$$

consists of exactly one member of each pair $A, A^c \in X \backslash (X' \cup X'')$. Thus $C_{P_1,\dots,P_n} = \cap_{A \in Y' \cup Y'' \cup Y_{P_1,\dots,P_n}} A$, where the set $Y' \cup Y'' \cup Y_{P_1,\dots,P_n}$ consists of exactly one member of each pair $A, A^c \in X$. So, as Σ is generated by X, $C_{P_1,\dots,P_n}$ is either an atom or is empty. Hence it suffices to show that $C_{P_1,\dots,P_n} \neq \emptyset$. Suppose the contrary. Then $Y' \cup Y'' \cup Y_{P_1,\dots,P_n}$ is inconsistent, hence has a minimal inconsistent subset Y. We distinguish two cases and derive a contradiction in each.

Case 1: there is a $B \in Y \cap Y_{P_1,\dots,P_n}$ with $\bar{A} \vdash\vdash^* B$. Consider any $B' \in Y \backslash \{B\}$. We have (i) not $\bar{A} \vdash\vdash^* B'$: otherwise, by $B' \vdash^* B^c$ we would have $\bar{A} \vdash\vdash^* B^c$, hence $B \in X'$, in contradiction to $B \in Y_{P_1,\dots,P_n}$. Further, as $\bar{A} \vdash\vdash^* B$ and $B \vdash^* (B')^c$, we have (ii) $\bar{A} \vdash\vdash^* (B')^c$. By (i) and (ii), and letting $A := (B')^c$, the event $A_{P_1,\dots,P_n}$ ($\in \{A, A^c\}$) is well-defined. As $Y_{P_1,\dots,P_n}$ contains $A_{P_1,\dots,P_n}$ ($\in \{A, A^c\}$), and contains $B' = A^c$ but not $(B')^c = A$, we must have $A_{P_1,\dots,P_n} = A^c$. So all i have $P_i(A) = 0$, i.e. all i have $P_i(B') = 1$. Since this holds for all $B' \in Y \backslash \{B\}$, all i have $P_i(\cap_{B' \in Y} B') = P_i(B)$. Hence, as Y is inconsistent, all i have $P_i(B) = 0$. Hence, $B_{P_1,\dots,P_n} = B^c$. So $B^c \in Y_{P_1,\dots,P_n}$, in contradiction to $B \in Y_{P_1,\dots,P_n}$.

Case 2: there is no $B \in Y \cap Y_{P_1,\dots,P_n}$ with $\bar{A} \vdash\vdash^* B$. Then all $B \in Y \cap Y_{P_1,\dots,P_n}$ take the form $A_{P_1,\dots,P_n} = A^c$, so that all i have $P_i(A) = 0$, i.e. all i have $P_i(B) = 1$. So, (*) all i have $P_i(\cap_{B \in Y} B) = P_i(\cap_{B \in Y \backslash Y_{P_1,\dots,P_n}} B)$. Now, we have either (i) $Y \subseteq Y_{P_1,\dots,P_n} \cup Y'$, or (ii) $Y \subseteq Y_{P_1,\dots,P_n} \cup Y''$, because otherwise there exist an $A' \in Y'$ and an $A'' \in Y''$, and we have $A' \vdash^* (A'')^c$, hence $\bar{A} \vdash\vdash^* (A'')^c$, a contradiction by $(A'')^c \in X''$. First suppose (i). Then $Y \backslash Y_{P_1,\dots,P_n} \subseteq Y'$, and so (*) implies that (**) all i have $P_i(\cap_{B \in Y} B) \geq P_i(\cap_{B \in Y'} B) = P_i(A')$. As by assumption $\tilde{P}_{P_1,\dots,P_n}(A') > 0$, there exists by (2) at least one i with $P_i(A') > 0$, hence by (**) with $P_i(\cap_{B \in Y} B) > 0$. So $\cap_{B \in Y} B \neq \emptyset$, i.e. Y is consistent, a contradiction. Similarly, under (ii) one can show that Y is consistent, a contradiction, q.e.d.

Now we define $P_{P_1,\dots,P_n}$ as the unique measure Σ that assigns the following measure to the atoms of Σ. If an atom takes the form in Claim 4, i.e. the form

$$B = C \cap (\cap_{A \in X: \bar{A} \vdash\vdash^* A \text{ and not } \bar{A} \vdash\vdash^* A^c} A_{P_1,\dots,P_n})$$

where $C \in \mathcal{A}(\tilde{\Sigma})$ and $\tilde{P}_{P_1,\dots,P_n}(C) > 0$, then we define its measure as

$$P_{P_1,\dots,P_n}(B) = \tilde{P}_{P_1,\dots,P_n}(C).$$

Any other atom has measure defined as zero.

Claim 5. $P_{P_1,...,P_n}$ extends $\tilde{P}_{P_1,...,P_n}$ (hence, is a probability measure).

It suffices to show that $P_{P_1,...,P_n}$ coincides with $\tilde{P}_{P_1,...,P_n}$ on $\mathcal{A}(\tilde{\Sigma})$. Consider any $C \in \mathcal{A}(\tilde{\Sigma})$. As Σ is a refinement of $\tilde{\Sigma}$, we have

$$P_{P_1,...,P_n}(C) = \sum_{B \in \mathcal{A}(\Sigma): B \subseteq C} P_{P_1,...,P_n}(B). \tag{3}$$

There are two cases.

Case 1: $\tilde{P}_{P_1,...,P_n}(C) = 0$. Then for all $B \in \mathcal{A}(\Sigma)$ with $B \subseteq C$ we have $P_{P_1,...,P_n}(B) = 0$ (by definition of $P_{P_1,...,P_n}$), and so by (3) we have $P_{P_1,...,P_n}(C) = 0 = \tilde{P}_{P_1,...,P_n}(C)$, as desired.

Case 2: $\tilde{P}_{P_1,...,P_n}(C) > 0$. Then, among all atoms $B \in \mathcal{A}(\Sigma)$ with $B \subseteq C$, there exists (by definition of $P_{P_1,...,P_n}$) exactly one with $P_{P_1,...,P_n}(B) > 0$ (namely $B = C \cap (\cap_{A \in X: \bar{A} \vdash\vdash^* A \text{ and not } \bar{A} \vdash\vdash^* A^c} A_{P_1,...,P_n}))$, and this B receives probability $P_{P_1,...,P_n}(B) = \tilde{P}_{P_1,...,P_n}(C)$. So by (3) we have $P_{P_1,...,P_n}(C) = \tilde{P}_{P_1,...,P_n}(C)$, q.e.d.

Claim 6. For all $A \in X$ such that $\bar{A} \vdash\vdash^* A$ and not $\bar{A} \vdash\vdash^* A^c$, $P_{P_1,...,P_n}(A)$ is 1 if some individual i has $P_i(A) > 0$, and 0 otherwise.

By definition of $P_{P_1,...,P_n}$, every atom of Σ that has positive probability is a subset of the event $\cap_{A \in X: \bar{A} \vdash\vdash^* A \text{ and not } \bar{A} \vdash\vdash^* A^c} A_{P_1,...,P_n}$, and so this event has probability 1. It follows that, for all $A \in X$ such that $\bar{A} \vdash\vdash^* A$ and not $\bar{A} \vdash\vdash^* A^c$, we have $P_{P_1,...,P_n}(A_{P_1,...,P_n}) = 1$, and hence

$$P_{P_1,...,P_n}(A) = \begin{cases} 1 & \text{if } A_{P_1,...,P_n} = A \text{, i.e. if } P_i(A) > 0 \text{ for some } i \\ 0 & \text{if } A_{P_1,...,P_n} = A^c \text{, i.e. if } P_i(A) = 0 \text{ for all } i, \end{cases}$$

q.e.d.

By Claim 5, we have constructed a well-defined pooling function $(P_1, ..., P_n) \mapsto P_{P_1,...,P_n}$. By (2) and Claims 5 and 6, we know its behaviour on the entire agenda X: the pooling function is independent with local decision criterion D_A given by

(i) the linear criterion D' if $A \in X'$,
(ii) the different linear criterion D'' if $A \in X''$,
(iii) a non-linear criterion $\hat{D}$ (taking everywhere except on $(0, ..., 0)$ the value 1) if $\bar{A} \vdash\vdash^* A$ but not $\bar{A} \vdash\vdash^* A^c$,
(iv) the non-linear criterion $1 - \hat{D}$ if not $\bar{A} \vdash\vdash^* A$ but $\bar{A} \vdash\vdash^* A^c$.

These decision criteria also ensure unanimity-preservation. To see that pooling is not neutral, it suffices to show that, of the four different types of events (i)-(iv), at least two occur. The latter is so because $\bar{A}$ is of type (i) or (iii) and because by assumption there exists an $A \in X$ such that not $\bar{A} \vdash\vdash^* A$, i.e. such that A has type (ii) or (iv). ■

Proof of Theorem 4. Our counterexample uses a set $\Omega := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ of (pairwise distinct) states ω_k, the σ-algebra $\Sigma := \{A : A \subseteq \Omega\}$ (the power set of

Ω), and the agenda $X := \{A \subseteq \Omega : |A| = 2\}$ (the set of binary events). As X is negation-closed and non-empty, it is indeed an agenda.

1. In this part of the proof, we show that X is pathconnected. Consider any events $A, B \in X$. We construct a path from A to B, by distinguishing three cases.

Case 1: $A = B$. Then the path is trivial, since $A \vdash^* A$ (take $Y = \emptyset$).

Case 2: A and B have exactly one world in common. We may then write $A = \{\omega_A, \omega\}$ and $B = \{\omega_B, \omega\}$ with $\omega_A, \omega_B, \omega$ pairwise distinct. We have $\{\omega_A, \omega\} \vdash^* \{\omega\}$ (take $Y = \{\{\omega, \omega'\}\}$, where ω' is the element of $\Omega \backslash \{\omega_A, \omega_B, \omega\}$) and $\{\omega\} \vdash^* \{\omega_B, \omega\}$ (take $Y = \emptyset$).

Case 3: : A and B have no world in common. We may then write $A = \{\omega_A, \omega'_A\}$ and $B = \{\omega_B, \omega'_B\}$ with $\omega_A, \omega'_A, \omega_B, \omega'_B$ pairwise distinct. We have $\{\omega_A, \omega'_A\} \vdash^* \{\omega_A, \omega_B\}$ (take $Y = \{\{\omega_A, \omega'_B\}\}$) and $\{\omega_A, \omega_B\} \vdash^* \{\omega_B, \omega'_B\}$ (take $Y = \{\{\omega_B, \omega'_A\}\}$).

2. In this part, we construct a pooling function $(P_1, ..., P_n) \mapsto P_{P_1,...,P_n}$ that is zero-preserving, neutral, but not linear. As an ingredient to the construction, consider first a linear pooling function $L : \mathcal{P}^n \to \mathcal{P}$. We show that L can be transformed into a non-linear pooling function that is still neutral and zero-preserving. We use an (arbitrary) fixed transformation $T : [0, 1] \to [0, 1]$ such that:

(i) $T(1 - x) = 1 - T(x)$ for all $x \in [0, 1]$ (hence $T(1/2) = 1/2$);

(ii) $T(0) = 0$ (hence by (i) $T(1) = 1$);

(iii) T is strictly concave on $[0, 1/2]$ (hence by (i) strictly convex on $[1/2, 1]$).

(Such a T indeed exists; e.g. $T(x) = 4(x - 1/2)^3 + 1/2$ for all $x \in [0, 1]$.)

We prove that for every probability measure $Q \in \mathcal{P}$ (thought of as the outcome of applying the linear pooling function L) there exist real numbers $p_k = p_k^Q$, $k = 1, 2, 3, 4$ (thought of as the new probabilities of the states ω_k, $k = 1, 2, 3, 4$, after transforming Q) such that:

(a) $p_1, p_2, p_3, p_4 \geq 0$ and $p_1 + p_2 + p_3 + p_4 = 1$;

(b) for all $A \in X$, $\sum_{k:\omega_k \in A} p_k = T(Q(A))$.

This completes the proof, because by (a) a pooling function $F : \mathcal{P}^n \to \mathcal{P}$, $(P_1, ..., P_n) \mapsto P_{P_1,...,P_n}$ can be defined by letting

$$P_{P_1,...,P_n}(A) := \sum_{k:\omega_k \in A} p_k^{L(P_1,...,P_n)} \text{ for all } A \in \Sigma,$$

which by (b) satisfies

$$P_{P_1,...,P_n}(A) = T(L(P_1, ..., P_n)(A)) \text{ for all } A \in X,$$

implying that F is neutral (as L is neutral), zero-preserving (as L is zero-preserving and $T(0) = 0$), and non-linear (as L is linear and T a non-linear transformation).

Let $Q \in \mathcal{P}^n$. For any $k \in \{1,2,3,4\}$, put $q^k := Q(\{\omega_k\})$; and for any $k,l \in \{1,2,3,4\}, k < l$, put $q_{kl} = Q(\{\omega_k, \omega_l\})$.

In order for numbers $p_1, ..., p_4$ to satisfy (b), they must satisfy the system

$$p_k + p_l = T(q_{kl}) \text{ for all } k,l \in \{1,2,3,4\} \text{ with } k < l.$$

Given $p_1 + p_2 + p_3 + p_4 = 1$, three of these six equalities are redundant. Indeed, suppose that $k,l \in \{1,2,3,4\}$, $k < l$, and define $k', l' \in \{1,2,3,4\}$, $k' < l'$, by $\{k', l'\} = \{1,2,3,4\} \backslash \{k,l\}$. By $p_k + p_l = 1 - p_{k'} - p_{l'}$ and $T(q_{kl}) = T(1 - q_{k'l'}) = 1 - T(q_{k'l'})$, the equality $p_k + p_l = T(q_{kl})$ is equivalent to $p_{k'} + p_{l'} = T(q^{k'l'})$. So (b) reduces (given $p_1 + p_2 + p_3 + p_4 = 1$) to the system

$$p_1 + p_2 = T(q_{12}),\ p_1 + p_3 = T(q_{13}),\ p_2 + p_3 = T(q_{23}).$$

We now solve this system of three linear equations in $(p_1, p_2, p_3) \in \mathbf{R}^3$. Write $t_{kl} := T(q_{kl})$ for all $k.l \in \{1,2,3,4\}$, $k < l$.

$$\begin{pmatrix} 1 & 1 & & t_{12} \\ 1 & & 1 & t_{13} \\ & 1 & 1 & t_{23} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & & t_{12} \\ & -1 & 1 & t_{13} - t_{12} \\ & & 2 & t_{23} + t_{13} - t_{12} \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} 1 & 1 & & t_{12} \\ & 1 & \text{-1} & t_{12} - t_{13} \\ & & 1 & \frac{t_{23}+t_{13}-t_{12}}{2} \end{pmatrix}.$$

So we have

$$\begin{aligned} p_3 &= \frac{t_{23} + t_{13} - t_{12}}{2}, \\ p_2 &= t_{12} - t_{13} + \frac{t_{23} + t_{13} - t_{12}}{2} = \frac{t_{12} + t_{23} - t_{13}}{2}, \\ p_1 &= t_{12} - \frac{t_{12} + t_{23} - t_{13}}{2} = \frac{t_{12} + t_{13} - t_{23}}{2}, \\ p_4 &= 1 - (p_1 + p_2 + p_3) = 1 - \frac{t_{12} + t_{13} + t_{23}}{2}. \end{aligned}$$

We have to show that the numbers $p_1, ..., p_4$ so-defined satisfy not only (b) and $p_1 + ... + p_4 = 1$ but also the remaining condition in (a), i.e. non-negativity. We do this by proving two claims.

Claim 1. $p_4 \geq 0$, i.e. $\frac{t_{12}+t_{13}+t_{23}}{2} \leq 1$.

We have to prove that $T(q_{12}) + T(q_{13}) + T(q_{23}) \leq 2$. Note that

$$q_{12} + q_{13} + q_{23} = q^1 + q^2 + q^1 + q^3 + q^2 + q^3 = 2(q^1 + q^2 + q^3) \leq 2.$$

We distinguish three cases.

Case 1: all of q_{12}, q_{13}, q_{23} are all $\geq 1/2$. Then by (i)-(iii) $T(q_{12}) + T(q_{13}) + T(q_{23}) \leq q_{12} + q_{13} + q_{23} \leq 2$, as desired.

Case 2: at least two of q_{12}, q_{13}, q_{23} are $< 1/2$. Then, again using (i)-(iii), $T(q_{12}) + T(q_{13}) + T(q_{23}) < 1/2 + 1/2 + 1 = 2$, as desired.

Case 3: exactly one of q_{12}, q_{13}, q_{23} is $< 1/2$. Suppose $q_{12} < 1/2 \leq q_{13} \leq q_{23}$ (otherwise just switch the roles of q_{12}, q_{13}, q_{23}). For all $\delta \geq 0$ such that $q_{13} - \delta, q_{23} + \delta \in [1/2, 1]$, the convexity of T on $[1/2, 1]$ implies that

$$\begin{aligned} T(q_{13}) &\leq \frac{1}{2}\left[T(q_{13} - \delta) + T(q_{23} + \delta)\right] \\ \text{and } T(q_{23}) &\leq \frac{1}{2}\left[T(q_{13} - \delta) + T(q_{23} + \delta)\right], \end{aligned}$$

so that (by adding these two inequalities)

$$T(q_{13}) + T(q_{23}) \leq T(q_{13} - \delta) + T(q_{23} + \delta).$$

This inequality may be applied to $\delta = 1 - q_{23}$, since

$$q_{13} - (1 - q_{23}) = (q_{13} + q_{23} + q_{12}) - q_{12} - 1 \leq 2 - q_{12} - 1 = 1 - q_{12} \in [1/2, 1];$$

which gives us

$$T(q_{13}) + T(q_{23}) \leq T(q_{13} - (1 + q_{23})) + T(1).$$

On the right hand side of this inequality, we have $T(1) = 1$ and, by $q_{13} - (1 + q_{23}) \leq 1 - q_{12}$ and T's increasingness, $T(q_{13} - (1 + q_{23})) \leq T(1 - q_{12}) = 1 - T(q_{12})$. So we obtain $T(q_{13}) + T(q_{23}) \leq 1 + 1 - T(q_{12})$, i.e. $T(q_{12}) + T(q_{13}) + T(q_{23}) \leq 2$, as desired.

Claim 2. $p_k \geq 0$ for all $k = 1, 2, 3$.

We only show that $p_1 \geq 0$, as the proofs for p_2 and p_3 are analogous. We have to prove that $t_{13} + t_{23} - t_{12} \geq 0$, i.e. that $T(q_{13}) + T(q_{23}) \geq T(q_{12})$, or equivalently that $T(q^1 + q^3) + T(q^2 + q^3) \geq T(q^1 + q^2)$. As T is an increasing function, it suffices to establish $T(q^1) + T(q^2) \geq T(q^1 + q^2)$. Again, we consider three cases.

Case 1: $q^1 + q^2 \leq 1/2$. Suppose $q^1 \leq q^2$ (otherwise the roles of q^1 and q^2 get swapped). For all $\delta \geq 0$ such that $q^1 - \delta, q^2 + \delta \in [0, 1/2]$, the concavity of T on $[0, 1/2]$ implies that

$$\begin{aligned} T(q^1) &\geq \frac{1}{2}\left[T(q^1 - \delta) + T(q^2 + \delta)\right] \\ \text{and } T(q^2) &\geq \frac{1}{2}\left[T(q^1 - \delta) + T(q^2 + \delta)\right], \end{aligned}$$

so that (by adding these inequalities)

$$T(q^1) + T(q^2) \geq T(q^1 - \delta) + T(q^2 + \delta)$$

Applying this to $\delta = q^1$ yields $T(q^1) + T(q^2) \geq T(0) + T(q^2 + q^1) = T(q^1 + q^2)$, as desired.

Case 2: $q^1 + q^2 > 1/2$ but $q^1, q^2 \leq 1/2$. By (i)-(iii),

$$T(q^1) + T(q^2) \geq q^1 + q^2 \geq T(q^1 + q^2),$$

as desired.

Case 3: $q^1 > 1/2$ or $q^2 > 1/2$. Suppose $q^2 > 1/2$ (otherwise swap q^1 and q^2 in the proof). Then $q^1 < 1/2$, as otherwise $q^1 + q^2 > 1$. Define $y := 1 - q^1 - q^2$. As also $y < 1/2$, an argument analogous to that in case 1 yields $T(q^1) + T(y) \geq T(q^1 + y)$, i.e. $T(q^1) + T(1 - q^1 - q^2) \geq T(1 - q^2)$. So, by (i), $T(q^1) + 1 - T(q^1 + q^2) \geq 1 - T(q^2)$, i.e. $T(q^1) + T(q^2) \geq T(q^1 + q^2)$. ■

One might wonder why the pooling function constructed in the proof of Theorem 4 violates implication-preservation – which it must do since Theorem 2 tells us that implication-preserving independent pooling functions must be linear (for non-simple, hence in particular for pathconnected agendas). Let Ω, Σ, X be as in the proof, and consider a profile with complete unanimity: all individuals i give ω_1 probability 0, each of ω_2, ω_3 probability 1/4, and hence ω_4 probability 1/2. As $\{\omega_1\}$ is the difference of two events in X (e.g. $\{\omega_1, \omega_2\} \backslash \{\omega_2, \omega_3\}$), implication-preservation would require the collective probability of ω_1 to be 0 too. But the collective probability of ω_1 is (in the notation of the proof) given by

$$p_1 = \frac{t_{12} + t_{13} - t_{23}}{2} = \frac{T(q_{12}) + T(q_{13}) - T(q_{23})}{2},$$

where q_{kl} is the collective probability of $\{\omega_k, \omega_l\}$ under a linear pooling function, so that q_{kl} equals the unanimous individual probability of $\{\omega_k, \omega_l\}$. So

$$p_1 = \frac{T(1/4) + T(1/4) - T(1/2)}{2} = T(1/4) - \frac{T(1/2)}{2},$$

which is strictly positive as T is strictly concave on $[0, 1/2]$ with $T(0) = 0$.