Judgment aggregation without full rationality: a summary

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Abstract

Several recent results on the aggregation of judgments over logically connected propositions show that, under certain conditions, dictatorships are the only independent (i.e., propositionwise) aggregation rules generating fully rational (i.e., complete and consistent) collective judgments. A frequently mentioned route to avoid dictatorships is to allow incomplete collective judgments. We show that this route does not lead very far: we obtain (strong) oligarchies rather than dictatorships if instead of full rationality we merely require that collective judgments be deductively closed, arguably a minimal condition of rationality (compatible even with empty judgment sets). We derive several characterizations of oligarchies and provide illustrative applications to Arrowian preference aggregation and Kasher and Rubinstein's group identification problem.

1 Introduction

Sparked by the "discursive paradox", the problem of "judgment aggregation" has recently received much attention. The "discursive paradox", of which Condorcet's famous paradox is a special case, consists in the fact that, if a group of individuals takes majority votes on some logically connected propositions, the resulting collective judgments may be inconsistent, even if all group members' judgments are individually consistent (Pettit 2001, extending Kornhauser and Sager 1986; List and Pettit 2004). A simple example is given in Table 1.

	a	b	$a \wedge b$
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
Majority	True	True	False

Table 1: A discursive paradox

Several subsequent impossibility results have shown that majority voting is not alone in its failure to ensure rational (i.e., complete and consistent) collective judgments when propositions are interconnected (List and Pettit 2002, Pauly and van Hees 2006, Dietrich 2006, Gärdenfors 2006, Nehring and Puppe 2002, 2005, van Hees forthcoming, Dietrich forthcoming, Mongin 2005, Dokow

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and Holzman 2005, Dietrich and List forthcoming-a). The generic finding is that, under the requirement of proposition-by-proposition aggregation (independence), dictatorships are the only aggregation rules generating complete and consistent collective judgments and satisfying some other conditions (which differ from result to result). This generic finding is broadly analogous to Arrow's theorem for preference aggregation. (Precursors to this recent literature are Wilson's 1975 and Rubinstein and Fishburn's 1986 contributions on abstract aggregation theory.)

A frequently mentioned escape route from this impossibility is to drop the requirement of complete collective judgments and thus to allow the group to make no judgment on some propositions. Examples of aggregation rules that ensure consistency at the expense of incompleteness are unanimity and certain supermajority rules (List and Pettit 2002, List 2004, Dietrich and List forthcoming-b).

The most forceful critique of the completeness requirement has been made by Gärdenfors (2006), in line with his influential theory of belief revision (e.g., Alchourron, Gärdenfors and Makinson 1985). Describing completeness as a "strong and unnatural assumption", Gärdenfors has argued that neither individuals nor a group need to hold complete judgments and that, in his opinion, "the [existing] impossibility results are consequences of an unnaturally strong restriction on the outcomes of a voting function". Gärdenfors has also proved the first and so far only impossibility result on judgment aggregation without completeness, showing that, under certain conditions, any aggregation rule generating consistent and deductively closed (but not necessarily complete) collective judgments, while not necessarily dictatorial, is weakly oligarchic.

In this paper, we continue this line of research and investigate judgment aggregation without the completeness requirement. We drop this requirement, first at the collective level and later at the individual level, replacing it with the weaker requirement of merely deductively closed judgments. Our results do not need the requirement of collective consistency. Under standard conditions on aggregation rules and the weakest possible assumptions about the agenda of propositions under consideration, we provide the first characterizations of (strong) oligarchies (without a default)[2] and the first characterization of the unanimity rule[3] (the only anonymous oligarchy). As corollaries, we also obtain new variants of several characterizations of dictatorships in the literature (using no consistency condition).

Our results strengthen Gärdenfors's oligarchy results in three respects. First, they impose weaker conditions on aggregation rules. Second, they show that strong and not merely weak oligarchies are implied by these conditions and fully

[2]For truth-functional agendas, Nehring and Puppe (2005) have characterized *oligarchies with a default*, which are distinct from the *(strong or weak) oligarchies* considered by Gärdenfors (2006) and in this paper. Oligarchies with a default by definition generate complete collective judgments.

[3]Again without a default, thus with possibly incomplete outcomes.

characterize strong oligarchies. Third, they do not require the logically rich and infinite agenda of propositions Gärdenfors assumes. They reinforce Gärdenfors's arguments, however, in showing that, under surprisingly mild conditions, we are restricted to oligarchic aggregation rules.

In judgment aggregation, one can distinguish between *impossibility results* (like Gärdenfors's results) and *characterizations of impossibility agendas* (like the present results and the results cited below). The former show that, for certain agendas of propositions, aggregation in accordance with certain conditions is impossible or severely restricted (e.g., to dictatorships or oligarchies). The latter characterize the precise class of agendas for which such an impossibility or restriction arises (and hence the class of agendas for which it does not arise). Characterizations of impossibility agendas have the merit of identifying precisely which kinds of decision problems are subject to the impossibility results in question and which are free from them. (Notoriously, preference aggregation problems are subject to most such impossibility results.) There has been much recent progress on such characterizations. Nehring and Puppe (2002) were the first to prove such results. Subsequent results have been derived by Dokow and Holzman (2005), Dietrich (forthcoming) and Dietrich and List (forthcoming-a). But so far all characterizations of impossibility agendas assume fully rational collective judgments. We here give the first characterizations of impossibility agendas without requiring complete (nor even consistent) collective judgments.

All proofs are given in Dietrich and List (2006).

2 The model

Consider a set of individuals $N = \{1, 2, \ldots, n\}$ ($n \geq 2$) seeking to make collective judgments on some logically connected propositions. To represent propositions, we introduce a logic, using Dietrich's (forthcoming) general logics framework (generalizing List and Pettit 2002, 2004). A *logic (with negation symbol* $\neg$*)* is a pair $(\mathbf{L}, \vDash)$ such that

(i) $\mathbf{L}$ is a non-empty set of formal expressions (*propositions*) closed under negation (i.e., $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$), and
(ii) $\vDash$ is a binary (*entailment*) relation ($\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L}$), where, for each $A \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $A \vDash p$ is read as "A entails p".

A set $A \subseteq \mathbf{L}$ is *inconsistent* if $A \vDash p$ and $A \vDash \neg p$ for some $p \in \mathbf{L}$, and *consistent* otherwise. Our results hold for any logic $(\mathbf{L}, \vDash)$ satisfying four minimal conditions;[4] this includes standard propositional, predicate, modal and

[4] L1 (self-entailment): For any $p \in \mathbf{L}$, $\{p\} \vDash p$. L2 (monotonicity): For any $p \in \mathbf{L}$ and any $A \subseteq B \subseteq \mathbf{L}$, if $A \vDash p$ then $B \vDash p$. L3 (completability): $\emptyset$ is consistent, and each consistent set $A \subseteq \mathbf{L}$ has a consistent superset $B \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$. L4 (non-paraconsistency): For any $A \subseteq \mathbf{L}$ and any $p \in \mathbf{L}$, if $A \cup \{\neg p\}$ is inconsistent then $A \vDash p$. In L4, the converse implication also holds given L1-L3. See Dietrich (forthcoming, Section 4) for the main properties of entailment and inconsistency under L1-L4.

conditional logics. For example, in standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \rightarrow b)$, and $\vDash$ satisfies $\{a, a \rightarrow b\} \vDash b$, $\{a\} \vDash a \vee b$, but not $a \vDash a \wedge b$.

A proposition $p \in \mathbf{L}$ is a *tautology* if $\{\neg p\}$ is inconsistent, and a *contradiction* if $\{p\}$ is inconsistent. A proposition $p \in \mathbf{L}$ is *contingent* if it is neither a tautology nor a contradiction. A set $A \subseteq \mathbf{L}$ is *minimal inconsistent* if it is inconsistent and every proper subset $B \subsetneq A$ is consistent.

The *agenda* is a non-empty subset $X \subseteq \mathbf{L}$, interpreted as the set of propositions on which judgments are to be made, where X can be written as $\{p, \neg p : p \in X^*\}$ for a set $X^* \subseteq \mathbf{L}$ of unnegated propositions. For notational simplicity, double negations within the agenda cancel each other out, i.e., $\neg\neg p$ stands for p.[5] In the example above, the agenda is $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$ in standard propositional logic. Informally, an agenda captures a particular decision problem.

An *(individual or collective) judgment set* is a subset $A \subseteq X$, where $p \in A$ means that proposition p is accepted (by the individual or group). Different interpretations of "acceptance" can be given. On the standard interpretation, to accept a proposition means to believe it, so that judgment aggregation is the aggregation of (binary) belief sets. On an entirely different interpretation, to accept a proposition means to desire it, so that judgment aggregation is the aggregation of (binary) desire sets.

A judgment set $A \subseteq X$ is

(i) *consistent* if it is a consistent set in $\mathbf{L}$,
(ii) *complete* if, for every proposition $p \in X$, $p \in A$ or $\neg p \in A$,
(iii) *deductively closed* if, for every proposition $p \in X$, if $A \vDash p$ then $p \in A$.

Note that the conjunction of consistency and completeness implies deductive closure, while the converse does not hold (Dietrich forthcoming, List 2004). Deductive closure can be met by "small", even empty, judgment sets $A \subseteq X$. Hence deductive closure is a much weaker requirement than "full rationality" (the conjunction of consistency and completeness). Let $\mathcal{C}$ be the set of all complete and consistent (and hence also deductively closed) judgment sets $A \subseteq X$. A *profile* is an n-tuple $(A_1, \ldots, A_n)$ of individual judgment sets.

A *(judgment) aggregation rule* is a function F that assigns to each admissible profile $(A_1, \ldots, A_n)$ a collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$. The set of admissible profiles is denoted $Domain(F)$.

Call F *universal* if $Domain(F) = \mathcal{C}^n$; call it *consistent*, *complete*, or *deductively closed* if it generates a consistent, complete, or deductively closed collective judgment set $A = F(A_1, \ldots, A_n)$ for every profile $(A_1, \ldots, A_n) \in Domain(F)$; call it *unanimity-respecting* if $F(A, ..., A) = A$ for all unanimous profiles $(A, ..., A) \in Domain(F)$; and call it *anonymous* if, for any profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$ that are permutations of each other,

[5] To be precise, when we use the negation symbol $\neg$ hereafter, we mean a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is unnegated and $\sim p := q$ if $p = \neg q$ for some q.

$F(A_1, \ldots, A_n) = F(A_1^*, \ldots, A_n^*)$.

Examples of aggregation rules are *majority voting*, where, for each $(A_1, ..., A_n) \in \mathcal{C}^n$, $F(A_1, ..., A_n) = \{p \in X : |\{i \in N : p \in A_i\}| > |\{i \in N : p \notin A_i\}|\}$ and a *dictatorship* of some individual $i \in N$, where, for each $(A_1, ..., A_n) \in \mathcal{C}^n$, $F(A_1, ..., A_n) = A_i$. Majority voting and dictatorships are each universal and unanimity-respecting. Majority voting is anonymous while dictatorships are not. But, as the "discursive paradox" shows, majority voting is not consistent (or deductively closed) (and it is complete if and only if n is odd), while dictatorships are consistent, complete and deductively closed. For some agendas X, so-called premise-based and conclusion-based aggregation rules can be defined.

The model can represent various realistic decision problems, including Arrowian preference aggregation problems and Kasher and Rubinstein's group identification problem, as illustrated in Sections 4 and 5.

3 Characterization results

Are there any appealing aggregation rules F if we allow incomplete outcomes? Our results share with previous results the requirement of *propositionwise aggregation*: the group "votes" independently on each proposition, as captured by the following condition.

Independence. For any $p \in X$ and any $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow p \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow p \in F(A_1^*, \ldots, A_n^*)$.

Interpretationally, independence requires the group judgment on any given proposition $p \in X$ to "supervene" on the individual judgments on p (List and Pettit forthcoming). This reflects a "local" notion of democracy, which could for instance be viewed as underlying direct democratic systems that are based on referenda on various propositions. If we require the group not only to vote independently on the propositions, but also to use the same voting method for each proposition (a neutrality condition), we obtain the following stronger condition.

Systematicity. For any $p, q \in X$ and any $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow q \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow q \in F(A_1^*, \ldots, A_n^*)$.

Some of our results require systematicity (and not just independence), and some also require the following responsiveness property.

Monotonicity. For any $(A_1, ..., A_n) \in Domain(F)$, we have $F(A_1^*, ..., A_n^*) = F(A_1, ..., A_n)$ for all $(A_1^*, ..., A_n^*) \in Domain(F)$ arising from $(A_1, ..., A_n)$ by replacing one A_i by $F(A_1, ..., A_n)$.

Monotonicity states that changing one individual's judgment set towards the present outcome (collective judgment set) does not alter the outcome.[6]

We call an aggregation rule F a *(strong) oligarchy* (dropping "strong" whenever there is no ambiguity) if it is universal and given by

$$F(A_1, .., A_n) = \cap_{i \in M} A_i \text{ for all profiles } (A_1, ..., A_n) \in \mathcal{C}^n, \tag{1}$$

where $M \subseteq N$ is fixed non-empty set (of *oligarchs*). A *weak oligarchy* is a universal aggregation rule F such that there exists a smallest winning coalition, i.e., a smallest non-empty set $M \subseteq N$ that satisfies (1) with "$=$" replaced by "$\supseteq$".[7] An oligarchy (respectively, weak oligarchy) accepts all (respectively, at least all) propositions unanimously accepted by the oligarchs.

Interesting impossibility results on judgment aggregation never apply to all agendas X (decision problems). Typically, impossibilities using the (strong) systematicity condition apply to most relevant agendas, while impossibilities using the (weaker) independence condition apply to a class of agendas that both includes and excludes many relevant agendas. Our present results confirm this picture.

We here use two weak agenda conditions (for our systematicity results) and one much stronger one (for our independence results). For any sets $Z \subseteq Y \subseteq X$, let $Y_{\neg Z}$ denote the set $(Y \backslash Z) \cup \{\neg p : p \in Z\}$, which arises from Y by negating the propositions in Z. The two weak conditions are the following.

(α) There is an inconsistent set $Y \subseteq X$ with pairwise disjoint subsets $Z_1, Z_2, \{p\}$ such that $Y_{\neg Z_1}$, $Y_{\neg Z_2}$ and $Y_{\neg \{p\}}$ are consistent.
(β) There is an inconsistent set $Y \subseteq X$ with disjoint subsets $Z, \{p\}$ such that $Y_{\neg Z}$, $Y_{\neg \{p\}}$ and $Y_{\neg (Z \cup \{p\})}$ are consistent.

These conditions are not *ad hoc*. As shown later, they are the weakest possible conditions needed for our results. If X is finite or the logic compact, (α) and (β) become equivalent to, respectively, the following standard conditions (see Dietrich and List 2006).

(i) There is a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$.
(ii) There is a minimal inconsistent set $Y \subseteq X$ such that $Y_{\neg Z}$ is consistent for some subset $Z \subseteq Y$ of even size (the *even-number negation* condition

[6] This is a judgment-set-wise monotonicity condition, which differs from a proposition-wise one (e.g., Dietrich and List 2005). Similarly, our condition of unanimity-respectance is judgment-set-wise rather than proposition-wise. One may consider this as an advantage, since a flavour of independence is avoided, so that the conditions in the characterisation are in the inutitive sense "orthogonal" to each other.

[7] The term "oligarchy" (without further qualification) refers to a strong oligarchy, whereas in Gärdenfors (2006) it refers to a weak one. A distinct oligarchy notion is Nehring and Puppe's (2005) "oligarchy with a default", which always generates complete collective judgments by reverting to a default on each pair $p, \neg p \in X$ on which the oligarchs disagree.

in Dietrich (forthcoming) and Dietrich and List (forthcoming-a), which for finite X is equivalent to Dokow and Holzman's (2005) *non-affineness* condition).

These conditions hold for most standard examples of judgment aggregation agendas X. For instance, if X contains propositions $a, b, a \wedge b$ as in the example of Table 1, then in (i) and (ii) we can take $Y = \{a, b, \neg(a \wedge b)\}$, where in (ii) $Z = \{a, b\}$. If X contains propositions $a, a \rightarrow b, b$ ("$\rightarrow$" could be a subjunctive implication) then in (i) and (ii) we can take $Y = \{a, a \rightarrow b, \neg b\}$, where in (ii) $Z = \{a, \neg b\}$. In Sections 4 and 5, we show that the conditions also hold for agendas representing Arrowian preference aggregation or Kasher and Rubinstein's group identification problem.

The stronger agenda condition, used in Theorem 2, is that of *path-connectedness*, a variant of Nehring and Puppe's (2002) *total blockedness* condition. For any $p, q \in X$, we write $p \vDash^* q$ (*p conditionally entails q*) if $\{p\} \cup Y \vDash q$ for some $Y \subseteq X$ consistent with p and with $\neg q$. For instance, for the agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$, we have $a \wedge b \vDash^* a$ (take $Y = \emptyset$) and $a \vDash^* \neg b$ (take $Y = \{\neg(a \wedge b)\}$). An agenda X is *path-connected* if, for every contingent $p, q \in X$, there exist $p_1, p_2, ..., p_k \in X$ (with $p = p_1$ and $q = p_k$) such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$.

The agenda $X = \{a, \neg a, b, \neg b, a \wedge b, \neg(a \wedge b)\}$ is *not* path-connected: for a negated proposition ($\neg a$ or $\neg b$ or $\neg(a \wedge b)$), there is no path to a non-negated proposition. By contrast, as discussed in Sections 4 and 5, the agendas for representing Arrowian preference aggregation problems or Kasher and Rubinstein's group identification problem are path-connected.

Theorem 1 *Let the agenda X satisfy (α) and (β).*

(a) *The oligarchies are the only universal, deductively closed, unanimity-respecting and systematic aggregation rules.*
(b) *Part (a) continues to hold if the agenda condition (β) is dropped and the aggregation condition of monotonicity is added.*

Theorem 2 *Let the agenda X satisfy path-connectedness and (β).*

(a) *The oligarchies are the only universal, deductively closed, unanimity-respecting and independent aggregation rules.*
(b) *Part (a) continues to hold if the agenda condition (β) is dropped and the aggregation condition of monotonicity is added.*

Proofs are given in the Appendix. Theorems 1 and 2 provide four characterizations of oligarchies. They differ in the conditions imposed on aggregation rules and the agendas permitted. Part (a) of Theorem 2 is perhaps the most surprising result, as it characterizes oligarchies on the basis of the logically weakest set of conditions on aggregation rules. We later apply this result to Arrowian preference aggregation problems and Kasher and Rubinstein's group identification problem.

In each characterization, adding the condition of anonymity eliminates all oligarchies except the *unanimity rule* (i.e., the oligarchy with $M = N$), and adding the condition of completeness eliminates all oligarchies except dictatorships (as defined above). So we obtain characterizations of the unanimity rule and of dictatorships.

Corollary 1 (a) *In each part of Theorems 1 and 2, the unanimity rule is the only aggregation rule satisfying the specified conditions and anonymity.*
(b) *In each part of Theorems 1 and 2, dictatorships are the only aggregation rules satisfying the specified conditions and completeness.*

Note that none of the characterizations of oligarchic, dictatorial or unanimity rules uses a collective consistency condition: consistency follows from the other conditions, as is seen from the consistency of oligarchic, dictatorial or unanimity rules.

As mentioned in the introduction, our results are related to (and strengthen) Gärdenfors's (2006) oligarchy results. We discuss the exact relationship in Section 6, when we relax the requirement of completeness not only at the collective level but also at the individual one.

Part (b) of Corollary 1 is also related to the characterizations of dictatorships by Nehring and Puppe (2002), Dokow and Holzman (2005) and Dietrich and List (forthcoming-a). To be precise, the dictatorship corollaries derived from parts (a) of Theorems 1 and 2 are variants (without a collective consistency condition) of Dokow and Holzman's (2005) and Dietrich and List's (forthcoming-a) characterizations of dictatorships.[8] The dictatorship corollaries derived from parts (b) of Theorems 1 and 2 are variants (again without a collective consistency condition) of Nehring and Puppe's (2002) characterizations of dictatorships.

As announced in the introduction, we seek to characterize impossibility agendas. While Theorems 1 and 2 establish the sufficiency of our agenda conditions for the present oligarchy results, we also need to establish their necessity. This is done by the next result. The proof consists in the construction of appropriate non-oligarchic counterexamples, given in the Appendix.[9]

Theorem 3 *Suppose $n \geq 3$ (and X contains at least one contingent proposition).*

(a) *If the agenda condition (β) is violated, there is a non-oligarchic (in fact, non-monotonic) aggregation rule that is universal, deductively closed, unanimity-respecting and systematic.*

[8] Our agenda conditions are, in the general case, at least as strong as those of the mentioned other dictatorship characterizations; but they are equivalent to them if X is finite or belongs to a compact logic (because then (β) reduces to a standard condition; see Section 3).

[9] Part (c) still holds for $n = 2$. It also follows from a rule specified by Nehring and Puppe (2002); our proof uses a simpler (and non-complete) rule.

(b) *If the agenda condition (α) is violated, there is a non-oligarchic aggregation rule that is universal, deductively closed, unanimity-respecting, systematic and monotonic.*
(c) *If the agenda is not path-connected, and is finite or belongs to a compact logic, there is a non-oligarchic (in fact, non-systematic) aggregation rule that is universal, deductively closed, unanimity-respecting, independent and monotonic.*

4 Application I: preference aggregation

We apply Theorem 2 to the aggregation of (strict) preferences, specifically to the case where a profile of fully rational individual preference orderings is to be aggregated into a possibly partial collective preference ordering.

To represent this aggregation problem in the judgment aggregation model, consider the *preference agenda* (Dietrich and List forthcoming-a; see also List and Pettit 2004), defined as $X = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K \text{ with } x \neq y\}$, where

(i) $\mathbf{L}$ is a simple predicate logic, with
  - a two-place predicate P (representing strict preference), and
  - a set of constants $K = \{x, y, z, ...\}$ (representing alternatives);

(ii) for each $S \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $S \vDash p$ if and only if $S \cup Z$ entails p in the standard sense of predicate logic, with Z defined as the set of rationality conditions on strict preferences.[10]

We claim that strict preference orderings can be formally represented as judgments on the preference agenda. Call a binary preference relation $\succ$ on K a *strict partial ordering* if it is asymmetric and transitive, and call $\succ$ a *strict ordering* if it is in addition connected. Notice that (i) the mapping that assigns to each strict partial ordering $\succ$ the judgment set $A = \{xPy, \neg yPx \in X : x \succ_i y\} \subseteq X$ is a bijection between the set of all strict partial orderings and the set of all consistent and deductively closed (but not necessarily complete) judgment sets; and (ii) the restriction of this mapping to strict orderings is a bijection between the set of all strict orderings and the set of all consistent and complete (hence deductively closed) judgment sets.

To apply Theorem 2, we observe that the preference agenda for three or more alternatives satisfies the agenda conditions of Theorem 2.

Lemma 1 *If $|K| \geq 3$, the preference agenda satisfies path-connectedness and (β).*[11]

[10] Z contains $(\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1)$ (asymmetry), $(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3)$ (transitivity), $(\forall v_1)(\forall v_2)(\neg\, v_1{=}v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1))$ (connectedness) and, for each pair of distinct contants $x, y \in K$, $\neg\, x{=}y$.

[11] Nehring (2003) has proved the path-connectedness result for the (weak) preference agenda.

Corollary 2 *For a preference agenda with* $|K| \geq 3$*, the oligarchies are the only universal, deductively closed (and also consistent), unanimity-respecting and independent aggregation rules.*

We have bracketed consistency since the result does not need the condition, although the interpretation offered above assumes it. In the terminology of preference aggregation, Corollary 2 shows that the oligarchies are the only preference aggregation rules with universal domain (of strict orderings) generating strict partial orderings and satisfying the weak Pareto principle and independence of irrelevant alternatives. Here an *oligarchy* is a preference aggregation rule such that, for each profile of strict orderings $(\succ_1, ..., \succ_n)$, the collective strict partial ordering $\succ$ is defined as follows: for any alternatives $x, y \in K$, $x \succ y$ if and only if $x \succ_i y$ for all $i \in M$, where $M \subseteq N$ is an antecedently fixed non-empty set of *oligarchs*.

This corollary is closely related to Gibbard's (1969) classic result showing that, if the requirement of transitive social orderings in Arrow's framework is weakened to that of quasi-transitive ones (requiring transitivity only for the strong component of the social ordering, but not for the indifference component), then oligarchies (suitably defined for the case of weak preference orderings) are the only preference aggregation rules satisfying the remaining conditions of Arrow's theorem. The relationship to our result lies in the fact that the strong component of a quasi-transitive social ordering is a strict partial ordering, as defined above.

5 Application II: group identification

Here we apply Theorem 2 to Kasher and Rubinstein's (1997) problem of "group identification". A set $N = \{1, 2, ..., n\}$ of individuals (e.g., a population) each make a judgment $J_i \subseteq N$ on which individuals in that set belong to a particular social group, subject to the constraint that at least one individual belongs to the group but not all individuals do (formally, each J_i satisfies $\varnothing \subsetneq J_i \subsetneq N$). The individuals then seek to aggregate their judgments $(J_1, ..., J_n)$ on who belongs to the social group into a resulting collective judgment J, subject to the same constraint ($\varnothing \subsetneq J \subsetneq N$). Thus Kasher and Rubinstein analyse the case in which the group membership status of all individuals must be settled definitively.

By contrast, we apply Theorem 2 to the case in which the membership status of individuals can be left undecided: i.e., some individuals are deemed members of the group in question, others are deemed non-members, and still others are left undecided with regard to group membership, subject to the very minimal "deductive closure" constraint that if all individuals except one are deemed non-members, then the remaining individual must be deemed a member, and if all individuals except one are deemed members, then the remaining individual must be deemed a non-member.

To represent this problem in our model (drawing on a construction in List 2006), consider the *group identification agenda*, defined as $X = \{a_1, \neg a_1, ..., a_n, \neg a_n\}$, where

(i) **L** is a simple propositional logic, with atomic propositions a_1, ..., a_n and the standard connectives $\neg$, $\wedge$, $\vee$;

(ii) for each $S \subseteq \mathbf{L}$ and each $p \in \mathbf{L}$, $S \vDash p$ if and only if $S \cup Z$ entails p in the the standard sense of propositional logic, where $Z = \{a_1 \vee ... \vee a_n, \neg(a_1 \wedge ... \wedge a_n)\}$.

Informally, a_j is the proposition that "individual j is a member of the social group", and $S \models p$ means that S implies p relative to the constraint that the disjunction of a_1, ..., a_n is true and their conjunction false. The mapping that assigns to each J (with $\varnothing \subsetneq J \subsetneq N$) the judgment set $A = \{a_j : j \in J\} \cup \{\neg a_j : j \notin J\} \subseteq X$ is a bijection between the set of all fully rational judgments in the Kasher and Rubinstein sense and the set of all consistent and complete judgment sets in our model. A merely deductively closed judgment set $A \subseteq X$ represents a judgment that possibly leaves the membership status of some individuals undecided, as outlined above and illustrated more precisely below.

To apply Theorem 2, we observe that the group identification agenda for three or more individuals ($n \geq 3$) satisfies the agenda conditions of Theorem 2.

Lemma 2 *If $n \geq 3$, the group identification agenda satisfies path-connectedness and (β).*

Corollary 3 *For a group identification agenda with $n \geq 3$, the oligarchies are the only universal, deductively closed (and consistent), unanimity-respecting and independent aggregation rules.*

In group identification terms, the oligarchies are the only group identification rules with universal domain generating possibly incomplete but deductively closed group membership judgments and satisfying unanimity and independence. Here an *oligarchy* is a group identification rule such that, for each profile $(J_1, ..., J_n)$ of fully rational individual judgments on group membership, the collective judgment is given as follows: the set of determinate group members is $\bigcap_{i \in M} J_i$, the set of determinate non-members is $\bigcap_{i \in M} (N \backslash J_i)$, and the set of individuals with undecided membership status is the complement of these two sets in N, where $M \subseteq N$ is an antecedently fixed non-empty set of *oligarchs*.[12]

[12] In fact, the set of individuals whose group membership status is to be decided need not coincide with the set of individuals who submit judgments on who is a member. More generally, the set N can make judgments on which individuals in another set K ($|K| \geq 3$) belong to a particular social group, subject to the constraint stated above. K can be infinite. Corollary 3 continues to hold since the corresponding group identification agenda (for a suitably adapted logic) still satisfies path-connectedness and (β). Interestingly, if K is infinite the agenda belongs to a non-compact logic.

6 The case of incomplete individual judgments

As argued by Gärdenfors (2006), it is natural to relax the requirement of completeness not only at the collective level, but also at the individual one. Do the above impossibilities disappear if individuals can withhold judgments on some or even all pairs $p, \neg p \in X$? Unfortunately, the answer to this question is negative, even if the conditions of independence or systematicity are weakened by allowing the collective judgment on a proposition $p \in X$ to depend not only on the individuals' judgments on p but also on those on $\neg p$. Such weaker independence or systematicity conditions are arguably more defensible than the standard conditions: $\neg p$ is intimately related to p, and thus individual judgments on $\neg p$ should be allowed to matter for group judgments on p. As the weakened conditions are equivalent to the standard ones under individual completeness, all the results in Section 3 continue to hold for the weakened independence and systematicity conditions.

Formally, let $\mathcal{C}^*$ be the set of all consistent and deductively closed (but not necessarily complete) judgment sets $A \subseteq X$, and call F *universal** if F has domain $(\mathcal{C}^*)^n$ (a superdomain of $\mathcal{C}^n$). An *oligarchy** is the universal* variant of an oligarchy as defined above.

Following Gärdenfors (2006), call F *weakly independent* if, for any $p \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow p \in A_i^*$ and $\neg p \in A_i \Leftrightarrow \neg p \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow p \in F(A_1^*, \ldots, A_n^*)$. Likewise, call F *weakly systematic* if, for any $p, q \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in Domain(F)$, if [for all $i \in N$, $p \in A_i \Leftrightarrow q \in A_i^*$ and $\neg p \in A_i \Leftrightarrow \neg q \in A_i^*$] then $p \in F(A_1, \ldots, A_n) \Leftrightarrow q \in F(A_1^*, \ldots, A_n^*)$.

We now give analogues of parts (a) of Theorems 1 and 2, proved in the Appendix.

Theorem 1* *Let the agenda X satisfy (α) and (β). The oligarchies* are the only universal*, deductively closed, unanimity-respecting and weakly systematic aggregation rules.*

Theorem 2* *Let the agenda X satisfy path-connectedness and (β). The oligarchies* are the only universal*, deductively closed, unanimity-respecting and weakly independent aggregation rules.*

In analogy with Theorems 1 and 2, these characterizations of oligarchies* do not contain a collective consistency condition (but require individual consistency). In each of Theorems 1* and 2*, adding the collective completeness requirement (respectively, anonymity) narrows down the class of aggregation rules to dictatorial ones (respectively, the unanimity rule), extended to the domain $(\mathcal{C}^*)^n$. So Theorems 1* and 2* imply characterizations of the latter rules on the domain $(\mathcal{C}^*)^n$. Note, further, that our applications of Theorem 2 to the preference and group identification agendas in Sections 4 and 5 can accommodate the case of incomplete individual judgments by using Theorem 2* instead of Theorem 2.

We can finally revisit the relationship of our results with Gärdenfors's results. Theorem 2, Corollary 1 and Theorem 2* strengthen Gärdenfors's oligarchy results. First, they do not require Gärdenfors's "social consistency" condition.[13] Second, they show that the conditions on aggregation rules imply (and in fact fully characterize) strong and not merely weak oligarchies (respectively, oligarchies*). Third, they weaken Gärdenfors's assumption that the agenda has the structure of an atomless Boolean algebra, replacing it with the weakest possible agenda assumption under which the oligarchy result holds, i.e., path-connectedness and (β).

Our results show that allowing incomplete judgments while preserving deductive closure and (weak) independence does not lead very far into possibility terrain. To obtain genuine possibilities, deductive closure must be relaxed or – perhaps better – independence must be given up in favour of non-propositionwise aggregation rules.

7 References

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[13] Gärdenfors's "social logical closure" is equivalent to our "deductive closure", where *entailment* in Gärdenfors' Boolean algebra agenda X should be defined as follows: a set $A \subseteq X$ *entails* $p \in X$ if and only if $(\wedge_{q \in A_0} q) \wedge \neg p$ is the contradiction for some finite $A_0 \subseteq A$.

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